**Hilbert functions and lex ideals**

*Jeffrey Mermin* *Irena Peeva*

Department of Mathematics, Cornell University, Ithaca, NY 14853, USA.

Abstract: We study Hilbert functions of graded ideals using lex ideals.

## 1. Introduction

A well-studied and important numerical invariant of a homogeneous ideal over a standard graded polynomial ring $S$ is its Hilbert function. It gives the sizes of the graded components of the ideal.

There are many papers on Hilbert functions or using them. In many of the recent papers and books, Hilbert functions are described using Macaualay's representation with binomials. Thus, the arguments consist of very clever computations with binomials. We have intentionally avoided computations with binomials. Macaulay's original idea in 1927 [Ma] is that *there exist highly structured monomial ideals - lex ideals - that attain all possible Hilbert functions.* In our proofs and in the open problems, we discuss the role of lex ideals. It seems to us that Problems 3.1.6 and 3.1.8 are very basic and natural if one is focused on the lex ideals instead of computations with binomials.

Throughout the paper $S = k[x_1, \ldots, x_n]$ is a polynomial ring over a field $k$ graded by $\deg(x_i) = 1$ for all $i$. Let $P = (x_1^{e_1}, \cdots, x_n^{e_n})$, with $e_1 \leq e_2 \leq \cdots \leq e_n \leq \infty$ (here $x_i^\infty = 0$) and set $W = S/P$. The Clements-Lindström Theorem [CL] characterizes

*1991 Mathematics Subject Classification:* 13F20.

*Keywords and Phrases:* Hilbert function, lexicographic ideals.

the possible Hilbert functions of graded ideals in the quotient ring $W$; Macaulay's Theorem [Ma] covers the particular case when $W = S$. In Section 2, we present an algebraic proof of the Clements-Lindström Theorem combining ideas of Bigatti [Bi], Clements and Lindström [CL], and Green [Gr]. The proof is based on the argument in [MP]. One of our main results is the Comparison Theorem 2.14 which was inspired by Green's Theorem. Note that the Comparison Theorem 2.14 holds in the ring $W$. As an immediate corollary we obtain the Generalized Green's Theorem 2.15. Green's Theorem 2.16 is over the ring $S$, and is just a particular case of Theorem 2.15. Theorem 2.16 was first proved by Green [Gr2] for linear forms, then it was extended to non-linear forms by Gasharov, Herzog, and Popescu [Ga, HP].

In Section 3, we raise problems and conjectures which are natural extensions of:

- Macaulay's Theorem and Clements-Lindström's Theorem
- Evans' Conjecture on lex-plus-powers ideals
- conjectures by Gasharov, Herzog, Hibi, and Peeva.

We also very briefly discuss Eisenbud-Green-Harris Conjecture. All the problems focus on the role of lex ideals.

By Macaulay's Theorem [Ma] lex sequences of monomials have the minimal possible growth of the Hilbert function. There exist many other monomial sequences with this property. The study of such sequences is started by Mermin [Me]; they are called lexlike sequences. In Section 4, we introduce lexlike ideals and prove an extension of Macaulay's Theorem for lexlike ideals. By Macaulay's Theorem, every Hilbert function is attained by a (unique up to reordering of the variables) lex ideal. One of our main results, Theorem 4.11, shows that it is also attained by (usually many) lexlike ideals; this is illustrated in Example 4.12. Furthermore, we extend to lexlike ideals the result of Bigatti, Hulett, Pardue, that lex ideals have the greatest graded Betti numbers among all ideals with a fixed Hilbert function. We show in Theorem 4.14 that lexlike ideals have the greatest graded Betti numbers among all ideals with a fixed Hilbert function.

In the last section 5, we discuss multigraded Hilbert functions and introduce multilex ideals.

**Acknowledgments.** The second author is partially supported by NSF.

## 2. The Clements-Lindström and Green's Theorems

Throughout this section we use the following notation. Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$ graded by $\deg(x_i) = 1$ for all $i$. Let $P = (x_1^{e_1}, \cdots, x_n^{e_n})$, with $e_1 \leq e_2 \leq \cdots \leq e_n \leq \infty$ (here $x_i^\infty = 0$) and set $W = S/P$. We denote by $W_d$ the $k$-vector space spanned by all monomials in $W$ of degree $d$. Denote $\mathbf{m} = (x_1, \ldots, x_n)_1$ the $k$-vector space spanned by the variables. We order the variables $x_1 > \ldots > x_n$, and we denote by $\succ_{lex}$ the homogeneous lexicographic order on the monomials. For a monomial $m$, set $\max(m) = \max\{i |\, x_i \text{ divides } m\}$.

We say that $A_d$ is a *$W_d$-monomial space* if it can be spanned by monomials of degree $d$. We denote by $\{A_d\}$ the set of monomials (non-zero monomials in $W_d$) contained in $A_d$. The cardinality of this set is $|A_d| = \dim_k A_d$. By $\mathbf{m}A_d$ we mean the $k$-vector subspace $\Big(\mathbf{m}\,(A_d)\Big)_{d+1}$ of $W_{d+1}$.

The *lex-segment* $\lambda_{d,p}$ of length $p$ in degree $d$ is defined as the $k$-vector space spanned by the lexicographically first (greatest) $p$ monomials in $W_d$. We say that $\lambda_d$ is a *lex-segment* in $W_d$ if there exists a $p$ such that $\lambda_d = \lambda_{d,p}$. For a monomial space $A_d$, we say that $\lambda_{d,|A_d|}$ is its *$W_d$-lexification.*

Compressed ideals were introduced by Clements and Lindström [CL]. They play an important role in the proof of the theorem.

**Definition 2.1.** An $W_d$-monomial space $C_d$ is called *$i$-compressed* if we have the disjoint union

$$\{C_d\} = \coprod_{0 \leq j \leq d} x_i^{d-j}\{L_j\}$$

and each $L_j$ is a lex-segment in $\big(W/x_i\big)_j$. We say that a $k$-vector space $C_d$ is *$W_d$-compressed* (or compressed) if it is a $W_d$-monomial space and is $i$-compressed for all $1 \leq i \leq n$. A monomial ideal $M$ in $W$ is called *compressed* if $M_d$ is compressed for all $d \geq 0$.

**Lemma 2.2.** *If $C_d$ is $i$-compressed in $W_d$, then $\mathbf{m}C_d$ is $i$-compressed in $W_{d+1}$.*

Another class of ideals useful in the proof are Borel ideals, defined as follows.

**Definition 2.3.** A monomial $m' \in W$ is said to be *in the big shadow* of a monomial

$m \in W$ if $m' = \dfrac{x_i\, m}{x_j}$ for some $x_j$ dividing $m$ and some $i \le j$. We say that a monomial space $B_d$ in $W_d$ is *Borel* if it contains all monomials in the big shadows of its monomial generators.

**Lemma 2.4.**
*(1) If a monomial space $C_d$ is compressed and $n \ge 3$, then $C_d$ is Borel.*
*(2) If $n \le 2$, then every monomial space is compressed.*

*Proof:* We will prove (1). Let $m \in \{C_d\}$ and $m'$ be a monomial in its big shadow. Hence $m' = \dfrac{x_i\, m}{x_j}$ for some $x_j$ dividing $m$ and some $i \le j$. There exists an index $1 \le q \le n$ such that $q \ne i, j$. Note that that $m$ and $m'$ have the same $q$-exponents. Since $C_d$ is $q$-compressed and $m' \succ_{lex} m$, it follows that $m' \in \{C_d\}$. Therefore, $C_d$ is Borel. □

For a $W_d$-monomial space $A_d$ set

$$r_{i,j}(A_d) = \Big|\, \{\, m \in \{A_d\} \mid \max(m) \le i \ \text{ and } \ x_i^j \text{ does not divide } m \,\} \,\Big|.$$

The following lemma is a generalization of a result by Bigatti [Bi].

**Lemma 2.5.** *If a monomial space $B_d$ is $W_d$-Borel, then*

$$\Big|\, \{\mathbf{m}B_d\} \,\Big| = \sum_{i=1}^{n} r_{i,e_i-1}(B_d)\,.$$

*Proof:* We will show that $\{\mathbf{m}B_d\}$ is equal to the set

$$\coprod_{i=1}^{n} x_i \,\{\, m \in \{B_d\} \,|\, \max(m) \le i \,\} \ \setminus\ \coprod_{i=1}^{n} x_i \left\{ m \in \{B_d\} \,\middle|\, \begin{array}{l} \max(m) = i \text{ and} \\ x_i^{e_i-1} \text{ divides } \ m \end{array} \right\}.$$

Denote by $\mathcal{P}$ the set above. Let $w \in B_d$. For $j \ge \max(w)$ we have that $x_j w \in \mathcal{P}$. Let $j < \max(w)$. Then $v = x_j \dfrac{w}{x_{\max(w)}} \in B_d$. So, $x_j w = x_{\max(w)} v \in \mathcal{P}$. □

We recall the definition of lex ideal:

**Definition 2.6.** Let $L$ be a monomial ideal in $W$ minimally generated by monomials

$l_1, \ldots, l_r$. We say that $L$ is *lex*, ( *lexicographic*), if the following property is satisfied:

$$\left.\begin{array}{c} m \text{ is a monomial in } \ W \\ m \succ_{lex} l_i \ \text{ and } \deg(m) = \deg(l_i), \text{ for some } 1 \le i \le r \end{array}\right\} \quad \Longrightarrow \quad m \in L\,.$$

**Lemma 2.7.** *If $L_d$ is an $W_d$-lex-segment, then $\mathbf{m}L_d$ is an $W_{d+1}$-lex-segment.*

**Example 2.8.** The ideal $(x_1x_2, x_1x_3, x_2x_3)$ is lex in $k[x_1, x_2, x_3]/(x_1^2, x_2^2, x_3^5)$.

**Lemma 2.9.** *If $L_d$ is a lex-segment, then it is Borel and $W_d$-compressed.*

The main work for proving a generalized Green's theorem is in the following lemma:

**Lemma 2.10.** *Let $C_d$ be an $n$-compressed Borel $W_d$-monomial space, and let $L_d$ be a lex-segment in $W_d$ with $|L_d| \le |C_d|$. For each $1 \le i \le n$ and each $1 \le j \le e_i$ we have*

$$r_{i,j}(L_d) \le r_{i,j}(C_d)\,.$$

*Proof:* Note that both $L_d$ and $C_d$ are $W_d$-Borel and $n$-compressed.

First, we consider the case $i = n$. Clearly, $r_{n,e_n}(L_d) = |L_d| \le |C_d| = r_{n,e_n}(C_d)$ (if $e_n = \infty$, then we consider $r_{n,d+1}$ here). We induct on $j$ decreasingly. Suppose that $r_{i,j+1}(L_d) \le r_{i,j+1}(C_d)$ holds by induction.

If $\{C_d\}$ contains no monomial divisible by $x_n^j$ then

$$r_{i,j}(L_d) \le r_{i,j+1}(L_d) \le r_{i,j+1}(C_d) = r_{i,j}(C_d)\,.$$

Suppose that $\{C_d\}$ contains a monomial divisible by $x_n^j$. Denote by $e = x_1^{b_1} \ldots x_n^{b_n}$, with $b_n \ge j$, the lex-smallest monomial in $C_d$ that is divisible by $x_n^j$. Let $0 \le q \le j-1$. Since $C_d$ is $W_d$-Borel, it follows that $c_q = x_{n-1}^{b_n-q}\dfrac{e}{x_n^{b_n-q}} \in C_d$. This is the lex-smallest monomial that is lex-greater than $e$ and $x_n$ divides it at power $q$. Let the monomial $a = x_1^{a_1} \ldots x_{n-1}^{a_{n-1}} x_n^q \in W_d$ be lex-greater than $e$. Since $C_d$ is $n$-compressed and $a$ is lex-greater (or equal) than $c_q$, it follows that $a \in C_d$.

For a monomial $u$, we denote by $x_n \notin u$ the property that $x_n^j$ does not divide $u$. By what we proved above, it follows that

$$\text{(2.11)} \qquad \Big|\{u \in \{C_d\} \,|\, x_n \notin u,\ u \succ_{lex} e\,\}\big| = \big|\{u \in \{W_d\} \,|\, x_n \notin u,\ u \succ_{lex} e\,\}\Big|\,.$$

Therefore,

$$\begin{aligned}
r_{i,j}(L_d) &= |\{u \in \{L_d\} \,|\, x_n \notin u,\ u \succ_{lex} e\}| + |\{u \in \{L_d\} \,|\, x_n \notin u,\ u \prec_{lex} e\}| \\
&\le |\{u \in \{W_d\} \,|\, x_n \notin u,\ u \succ_{lex} e\}| + |\{u \in \{L_d\} \,|\, x_n \notin u,\ u \prec_{lex} e\}| \\
&\le |\{u \in \{W_d\} \,|\, x_n \notin u,\ u \succ_{lex} e\}| + |\{u \in \{L_d\} \,|\ u \prec_{lex} e\}| \\
&\le |\{u \in \{W_d\} \,|\, x_n \notin u,\ u \succ_{lex} e\}| + |\{u \in \{C_d\} \,|\ u \prec_{lex} e\}| \\
&= |\{u \in \{W_d\} \,|\, x_n \notin u,\ u \succ_{lex} e\}| + |\{u \in \{C_d\} \,|\, x_n \notin u,\ u \prec_{lex} e\}| \\
&= |\{u \in \{C_d\} \,|\, x_n \notin u,\ u \succ_{lex} e\}| + |\{u \in \{C_d\} \,|\, x_n \notin u,\ u \prec_{lex} e\}| \\
&= r_{i,j}(C_d)\,;
\end{aligned}$$

for the third inequality we used the fact that $L_d$ is a lex-segment in $W_d$ with $|L_d| \le |C_d|$; for the equality after that we used the definition of $e$; for the next equality we used (2.11). Thus, we have the desired inequality in the case $i = n$.

In particular, we proved that

$$r_{n,1}(L_d) \le r_{n,1}(C_d)\,. \tag{2.12}$$

Finally, we prove the lemma for all $i < n$. Both $\{C_d/x_n\}$ and $\{L_d/x_n\}$ are lex-segments in $W_d/x_n$ since $C_d$ is $n$-compressed. By (2.12) the inequality $r_{n,1}(L_d) \le r_{n,1}(C_d)$ holds, and it implies the inclusion $\{C_d/x_n\} \supseteq \{L_d/x_n\}$. The desired inequalities follow since

$$\begin{aligned}
r_{i,j}(C_d) &= r_{i,j}\big(C_d/(x_{i+1},\dots,x_n)\big) \\
r_{i,j}(L_d) &= r_{i,j}\big(L_d/(x_{i+1},\dots,x_n)\big)\,.
\end{aligned}$$

□

Let $B_d$ be a Borel monomial space in $W_d$. Set $z = x_n$ and $\mathbf{n} = (x_1,\dots,x_{n-1})_1 = \mathbf{m}/z$. We have the disjoint union

$$\{B_d\} = \coprod_{0\le j\le d} z^{d-j}\{U_j\}$$

where each $U_j$ is a monomial space in $W/z$. Let $F_j$ be the lexification of the space $U_j$ in $W/z$. Consider the $W_d$-monomial space $T_d$ defined by

$$\{T_d\} = \coprod_{0\le j\le d} z^{d-j}\{F_j\}\,.$$

Clearly, $|T_d| = |B_d|$. We call $T_d$ the *$n$-compression* of $B_d$.

**Lemma 2.13.** *Let $B_d$ be a Borel monomial space in $W_d$. Its $n$-compression $T_d$ is Borel.*

*Proof:* Consider the disjoint unions

$$\{B_d\} = \coprod_{0\le j\le d} z^{d-j}\{U_j\}$$

$$\{T_d\} = \coprod_{0\le j\le d} z^{d-j}\{F_j\}\,.$$

Since $B_d$ is Borel, it follows that $\mathbf{n}U_j \subseteq U_{j+1}$. Since $|F_j| = |U_j|$, we can apply Theorem 2.18(1) by induction on the number of the variables, and it follows that $|\mathbf{n}F_j| \le |\mathbf{n}U_j| \le |U_{j+1}| = |F_{j+1}|$. As both $\mathbf{n}F_j$ and $F_{j+1}$ are lex-segments, we conclude that $\mathbf{n}F_j \subseteq F_{j+1}$. If $x_n^{d-j}m$ is a monomial in $T_d$ and $m \in F_j$, then for each $1 \le i < n$ we have that $x_i m \in \mathbf{n}F_j \subseteq F_{j+1}$, so $x_n^{d-j-1}x_i m \in T_d$. If $x_p$ divides $m$, then for each $1 \le q \le p$ we have that $\frac{x_q m}{x_p} \in F_j$ since $F_j$ is lex. Thus, $T_d$ contains all the monomials in the big shadows of its monomials. We have proved that $T_d$ is Borel. $\square$

**Comparison Theorem 2.14.** *Let $B_d$ be a Borel monomial space in $W_d$. Let $L_d$ be a lex-segment in $W_d$ with $|L_d| \le |B_d|$. The following inequalities hold:*

$$r_{i,j}(L_d) \le r_{i,j}(B_d)\,.$$

*for each $1 \le i \le n$ and each $1 \le j \le e_i$.*

*Proof:* We prove the inequalities by decreasing induction on the number of variables $n$. Let $T_d$ be the $n$-compression of $B_d$. Since $T_d$ is Borel and $n$-compressed by Lemma 2.13, we can apply Lemma 2.10 and we get

$$r_{i,j}(L_d) \le r_{i,j}(T_d)$$

for each $1 \le i \le n$ and each $1 \le j \le e_i$. It remains to compare $r_{i,j}(T_d)$ and $r_{i,j}(B_d)$. For $i = n$, we have equalities $r_{n,j}(T_d) = r_{n,j}(B_d)$. Let $i < n$. Then $r_{i,j}(T_d) = r_{i,j}(T_d/x_n)$ and $r_{i,j}(B_d) = r_{i,j}(B_d/x_n)$, where $T_d/x_n = L_d$ is a lex-segment and $B_d/x_n = U_d$ is Borel. So, by induction the desired inequalities hold. $\square$

**Generalized Green's Theorem 2.15.** *Let $B_d$ be a Borel monomial space in $W_d$. Let $L_d$ be a lex-segment in $W_d$ with $|L_d| \leq |B_d|$. The following inequalities hold:*

$$\dim\Big(W_d/(L_d \oplus \mathbf{n}^{(d-j)}x_n^j)\Big) \geq \dim\Big(W_d/(B_d \oplus \mathbf{n}^{(d-j)}x_n^j)\Big)$$

*for each each $1 \leq j \leq e_n,\ j \leq d$.*

*Proof:* Note that the desired inequality is equivalent to

$$r_{n,j}(L_d) \leq r_{n,j}(B_d)\,.$$

It holds by Theorem 2.14. □

The following result is a straightforward corollary of Theorem 2.15 since $x_n^j$ is a generic form for every Borel ideal in $S$.

**Green's Theorem 2.16.** *Let $B_d$ be a Borel monomial space in $S_d$. Let $L_d$ be a lex-segment in $S_d$ with $|L_d| \leq |B_d|$. Let $g$ be a generic homogeneous form of degree $j \geq 1$. The following inequality holds:*

$$\dim\Big(S_d/(L_d \oplus \mathbf{m}^{(d-j)}g)\Big) \geq \dim\Big(S_d/(B_d \oplus \mathbf{m}^{(d-j)}g)\Big)\,.$$

**Remark 2.17.** Theorem 2.16 in the case when $j = 1$ was proved by Green [Gr2]. Theorem 2.16 in the case when $j > 1$ was proved by Gasharov, Herzog, and Popescu [Ga, HP]. Theorem 2.15 in the case when $j = 1$ was proved by Gasharov [Ga2, Theorem 2.1].

We are ready to prove Macaulay's Theorem [Ma] which characterizes the possible Hilbert functions of graded ideals in $S$. There are several different proofs of this theorem, cf. Green [Gr].

**Macaulay's Theorem 2.18.** *The following two properties are equivalent, and they hold:*

*(1) Let $A_d$ be a $S_d$-monomial space and $L_d$ be its lexification in $S_d$. Then $|\,\mathbf{m}L_d\,| \leq |\,\mathbf{m}A_d\,|$.*

*(2) For every graded ideal $J$ in $S$ there exists a lex ideal $L$ with the same Hilbert function.*

*Proof:* First, we will prove that (1) holds. Since $A_d$ and $L_d$ are monomial spaces, (1) does not depend on the field $k$. Thus, we can replace the field if necessary and assume that $k$ has characteristic zero. This makes it possible to use Gröbner basis theory to reduce to the Borel case, cf. [Ei, Chapter 15]. We obtain a Borel $S_d$-monomial space $B_d$ such that $|B_d| = |A_d|$ and $|\mathbf{m}B_d| \le |\mathbf{m}A_d|$. For a $S_d$-monomial space $D_d$ set $t_i(D_d) = r_{i+1,1}(D_d) = \Big|\{\, m \in \{D_d\} \mid \max(m) \le i\,\}\Big|$. We apply Lemma 2.5 to conclude that

$$\Big|\{\mathbf{m}B_d\}\Big| = \sum_{i=1}^{n} t_i(B_d) \quad \text{and} \quad \Big|\{\mathbf{m}L_d\}\Big| = \sum_{i=1}^{n} t_i(L_d)\,.$$

Finally, we apply Theorem 2.14 and get the inequality $\big|\{\mathbf{m}L_d\}\big| \le \big|\{\mathbf{m}B_d\}\big|$. We proved (1).

Now, we prove that (1) and (2) are equivalent. Clearly, (2) implies (1). We assume that (1) holds and will prove (2). We can assume that $J$ is a monomial ideal by Gröbner basis theory. For each $d \ge 0$, let $L_d$ be the lexification of $J_d$. By (1), it follows that $L = \oplus_{d\ge 0} L_d$ is an ideal. By construction, it is a lex-ideal and has the same Hilbert function as $J$ in all degrees. $\square$

We continue with the proof of Clements-Lindström Theorem.

**Lemma 2.19.** *Let $A_d$ be a $W_d$-monomial space. There exists a compressed monomial space $C_d$ in $W_d$ such that $|C_d| = |A_d|$ and $|\mathbf{m}C_d| \le |\mathbf{m}A_d|$.*

*Proof:* Suppose that $A_d$ is not $i$-compressed. Set $z = x_i$.

We have the disjoint union

$$\{A_d\} = \coprod_{0\le j\le d} z^{d-j}\{U_j\}$$

where each $U_j$ is a monomial space in $W/z$. Let $F_j$ be the lexification of the space $U_j$ in $W/z$. Consider the $W_d$-monomial space $T_d$ defined by

$$\{T_d\} = \coprod_{0\le j\le d} z^{d-j}\{F_j\}\,.$$

Clearly, $|T_d| = |A_d|$. We will prove that

$$|\mathbf{m}T_d| \le |\mathbf{m}A_d|\,.$$

We have the disjoint unions

$$\{\mathbf{m}A_d\} = \coprod_{0\le j\le d} z^{d-j+1}\{U_j + \mathbf{n}U_{j-1}\}$$

$$\{\mathbf{m}T_d\} = \coprod_{0\le j\le d} z^{d-j+1}\{F_j + \mathbf{n}F_{j-1}\}\,,$$

where $\mathbf{n} = \mathbf{m}/z$. We will show that

$$|F_j + \mathbf{n}F_{j-1}| = \max\Big\{|F_j|,\, |\mathbf{n}F_{j-1}|\Big\} \le \max\Big\{|U_j|,\, |\mathbf{n}U_{j-1}|\Big\} \le |U_j + \mathbf{n}U_{j-1}|\,.$$

The first equality above holds because both $F_j$ and $\mathbf{n}F_{j-1}$ are $(W/z)_j$-lex-segments, so $F_j + \mathbf{n}F_{j-1}$ is the longer of these two lex-segments. The last inequality is obvious. The middle inequality holds since by construction $F_{j-1}$ is the lexification of $U_{j-1}$, so $|F_{j-1}| = |U_{j-1}|$ and by induction on the number of variables we can apply Theorem 2.20(1) to the ring $W/z$.

Thus, $|F_j + \mathbf{n}F_{j-1}| \le |U_j + \mathbf{n}U_{j-1}|$. Multiplication by $z^{d-j+1}$ is injective if $d - j + 1 \le e_i - 1$ and is zero otherwise, therefore we conclude that

$$\Big|z^{d-j+1}(F_j + \mathbf{n}F_{j-1})\Big| \le \Big|z^{d-j+1}(U_j + \mathbf{n}U_{j-1})\Big|\,.$$

This implies the desired inequality $|\mathbf{m}T_d| \le |\mathbf{m}A_d|$.

Note that $\{T_d\}$ is greater lexicographically than $\{A_d\}$ (here "lexicographically greater" means that we order the monomials in $\{T_d\}$ and $\{A_d\}$ lexicographically, and then compare the two ordered sets lexicographically). If $T_d$ is not compressed, we can apply the argument above. After finitely many steps in this way, the process must terminate because at each step we construct a lex-greater monomial space. Thus, after finitely many steps, we reach a compressed monomial space. $\square$

The Clements and Lindström Theorem [CL] is:

**Clements and Lindström's Theorem 2.20.** *The following two properties are equivalent, and they hold:*

*(1) Let $A_d$ be a $W_d$-monomial space and $L_d$ be its lexification in $W_d$. Then $|\,\mathbf{m}L_d\,| \le |\,\mathbf{m}A_d\,|$.*

*(2) For every graded ideal $J$ in $W$ there exists a lex ideal $L$ with the same Hilbert function.*

*Proof:* First, we will prove that (1) holds. The theorem clearly holds if $n = 1$. An easy calculation shows that the theorem holds, provided $n = 2$ and we do not have $e_2 \le d+1 < e_1$. By the assumption on the ordering of the exponents, it follows that the theorem holds for $n = 2$.

Suppose that $n \ge 3$. First, we apply Lemma 2.19 to reduce to the compressed case. We obtain a compressed $W_d$-monomial space $C_d$ such that $|C_d| = |A_d|$ and $|\mathbf{m}C_d| \le |\mathbf{m}A_d|$. Both $L_d$ and $C_d$ are $(S/P)_d$-compressed. We apply Lemma 2.5 to conclude that

$$\Big|\{\mathbf{m}C_d\}\Big| = \sum_{i=1}^{n} r_{i,e_i-1}(C_d)$$

$$\Big|\{\mathbf{m}L_d\}\Big| = \sum_{i=1}^{n} r_{i,e_i-1}(L_d).$$

Finally, we apply Lemma 2.10 and obtain the inequality $\Big|\{\mathbf{m}L_d\}\Big| \le \Big|\{\mathbf{m}C_d\}\Big|$. We proved (1).

Now, we prove that (1) and (2) are equivalent. CLearly, (2) implies (1). We assume that (1) holds and will prove (2). We can assume that $J$ is a monomial ideal by Gröbner basis theory. For each $d \ge 0$, let $L_d$ be the lexification of $J_d$. By (1), it follows that $L = \oplus_{d\ge 0} L_d$ is an ideal. By construction, it is a lex-ideal and has the same Hilbert function as $J$ in all degrees. □

Lexicographic ideals are highly structured and it is easy to derive the inequalities characterizing their possible Hilbert functions.

## 3. Open Problems

Throughout this section, $M$ is monomial ideal in $S$.

### 3.1. Hilbert functions in quotient rings.

We focus on the problem to identify rings, other than $S$ and $S/P$, where Macaulay's and Clements-Lindström's theorems hold. First, we introduce the necessary definitions.

**Definition 3.1.1.** A homogeneous ideal of $S/M$ is *lexifiable* if there exists a lex ideal with the same Hilbert function. We say that $M$ and $S/M$ are *Macaulay-Lex* if every homogeneous ideal of $S/M$ is lexifiable.

The following problem is a natural extension of Macaulay's and Clements-Lindström's Theorems:

**Problem 3.1.2.** *Identify monomial ideals which are Macaulay-Lex.*

Macaulay's Theorem [Ma] says that 0 is Macaulay-Lex. Clements-Lindström's Theorem [CL] says that $(x_1^{e_1}, \cdots, x_n^{e_n})$ is Macaulay-Lex when $e_1 \leq \cdots \leq e_n \leq \infty$. In [MP], we prove the following theorems in this direction:

**Theorem 3.1.3.** *Let $M$ be Macaulay-Lex and $L$ be lex. Then $M+L$ is Macaulay-Lex.*

**Theorem 3.1.4.** *Let $S/M$ be Macaulay-Lex. Then $(S/M)[y]$ is Macaulay-Lex.*

Macaulay-Lex ideals appear to be rare, however. For example, Shakin [Sh] has recently shown that a Borel ideal $M$ is Macaulay-Lex if and only if it is *piecewise lex*, that is, if $M$ may be written $M = \sum L_i$ with $L_i$ generated by a lex segment of $k[x_1, \cdots, x_i]$.

It is easy to construct examples like [MP, Example 2.13], where a given Hilbert function is not attained by any lexicographic ideal in the degrees of the minimal generators of $M$. This suggests that our definitions should be relaxed somewhat. In [MP] we introduce the following definition:

**Definition 3.1.5.** We say that $M$ and $S/M$ are *pro-lex above* $q$ if every homogeneous ideal of $S/M$ generated in degrees $\geq q$ is lexifiable. Let $d$ be the maximal degree of a minimal generator of $M$. We say that $M$ and $S/M$ are *pro-lex* if they are pro-lex above $d$.

We have the following variation of Problem 3.2:

**Problem 3.1.6.** *Identify monomial ideals which are pro-lex.*

As a first step in this direction, we show in [MP]:

**Theorem 3.1.7.** *Let $K$ be a compressed monomial ideal, and let $d$ be the maximal degree of a minimal monomial generator of $K$. If $n = 2$, assume that $K$ is lex. Let*

*$P = (x_1^{e_1}, \cdots, x_n^{e_n})$, where $e_1 \leq \cdots e_n \leq \infty$ (here $x_i^\infty = 0$). The ideal $K + P$ is pro-lex above $d$.*

It is natural to try to extend to non-monomial ideals:

**Problem 3.1.8.** *Find other graded rings where the notion of lex ideal makes sense and which are pro-lex.*

Of particular interest are the coordinate rings of projective toric varieties. Toric varieties are an important class of varieties which occur at the intersection of Algebraic Geometry, Commutative Algebra, and Combinatorics. They might provide many examples of interesting rings in which all Hilbert functions are attained by lex ideals.

**Problem 3.1.9.** *Find projective toric rings which are pro-lex (or Macaulay-Lex).*

The coordinate rings of toric varieties admit a natural multigraded structure which refines the usual grading and which yields a multigraded Hilbert function; this is studied in Section 5.

### 3.2. The Eisenbud-Green-Harris Conjecture.

The most exciting currently open conjecture on Hilbert functions is given by Eisenbud, Green, and Harris in [EGH1, EGH2]. The conjecture is wide open.

**Conjecture 3.2.1.** *Let $N$ be a homogeneous ideal containing a homogeneous regular sequence in degrees $e_1 \leq \cdots \leq e_r$. There is a monomial ideal $T$ such that $N$ and $T + (x_1^{e_1}, \cdots, x_r^{e_r})$ have the same Hilbert function.*

The original conjecture differs from 3.2.1 in the following two minor aspects:

- In the original conjecture $r = n$.
- The original conjecture gives a numerical characterization of the possible Hilbert functions of $N$. It is well known that this numerical characterization is equivalent to the existence of a lex ideal $L$ such that $L + (x_1^{e_1}, \cdots, x_r^{e_r})$ has the same Hilbert function as $N$. By Clements-Lindström's Theorem, this is equivalent to Conjecture 3.2.1.

### 3.3. Betti numbers.

The study of Hilbert functions is often closely related to the study of free resolutions.

We focus on problems based on the idea that the lex ideal has the greatest Betti numbers among all ideals with a fixed Hilbert function.

**Conjecture 3.3.1.** *Let $k$ be an infinite field (possibly, one should also assume that $k$ has characteristic* 0*). Suppose that $S/M$ is pro-lex above $d$, $J$ is a homogeneous ideal in $S/M$, generated in degrees $\geq d$, and $L$ is the lex ideal with the same Hilbert function as $J$. Then:*
(1) *The Betti numbers of $J$ over $R$ are less than or equal to those of $L$.*
(2) *The Betti numbers of $J + M$ over $S$ are less than or equal to those of $L + M$.*

Note that the first part of the conjecture is about infinite resolutions (unless $M$ is generated by linear forms), whereas the second part is about finite ones.

In the case $M = 0$, Conjecture 3.3.1 holds by a result of Bigatti [Bi], Hulett [Hu], and Pardue [Pa]. Also, Conjecture 3.3.1(1) holds by a result of Aramova, Herzog, and Hibi [AHH] over an exterior algebra. Furthermore, Conjecture 3.3.1(2) was inspired by work of Graham Evans and his conjecture, cf. [FR]:

**Conjecture 3.3.2.** (Evans) *Suppose that a homogeneous ideal $I$ contains a regular sequence of homogeneous elements of degrees $a_1, \dots, a_n$ in $S$. Suppose that there exists a lex-plus-powers ideal $L$ with the same Hilbert function as $I$. Then the Betti numbers of $L$ are greater than or equal to those of $I$.*

Conjecture 3.3.2 was inspired by the Eisenbud-Green-Harris Conjecture 3.2.1.

When the regular sequence in Conjecture 3.3.2 consists of powers of the variables, Conjecture 3.3.2 coincides with Conjecture 3.3.1(2). Also, in the case when $M$ is generated by powers of the variables, Conjecture 3.3.1(1) coincides with a conjecture of Gasharov, Hibi, and Peeva [GHP], and in the case when $M$ is generated by squares of the variables Conjecture 3.3.1(2) coincides with a conjecture of Herzog and Hibi.

**Remark 3.3.3.** It is natural to wonder whether Conjecture 3.3.1 should have part (3) that states that the Betti numbers of $J$ over $S$ are less or equal to those of $L$. There is a counterexample in [GHP]: take $J = (x^2, y^2)$ in $k[x, y]/(x^3, y^3)$ and $L = (x^2, xy)$, then the graded Betti numbers of $L$ over $S$ are not greater or equal to those of $J$ over $S$. It should be noticed that $J$ and $L$ do not have the same Hilbert function as ideals in $S$.

## 4. Lex-like ideals

In this section we work over the polynomial ring $S = k[x_1, \cdots, x_n]$. Macaulay's Theorem [Ma] has the following two equivalent formulations (given in Theorem 2.18).

**Theorem 4.1.** *Let $A_d$ be a monomial space in degree $d$ and $L_d$ be the space spanned by a lex segment in degree $d$ such that $|A_d| = |L_d|$. Then $|\,\mathbf{m}L_d\,| \leq |\,\mathbf{m}A_d\,|$.*

**Theorem 4.2.** *For every graded ideal $J$ in $S$ there exists a lex ideal $L$ with the same Hilbert function.*

The goal of this section is to show that a generalization of Macaulay's Theorem holds for ideals generated by initial segments of lexlike sequences. Lexlike sequences were discovered by Mermin in [Me]; we recall the definition.

**Definition 4.3.** A *monomial sequence* (of a fixed degree $d$) is a sequence $X_d$ of all the monomials of $S = k[x_1, \cdots, x_n]$ of degree $d$. We denote by $X_d(i)$ the monomial space generated by the first $i$ monomials in $X_d$. We say that $X_d$ is *lexlike* if, for every $i$, and for every vector space $V$ generated by $i$ monomials of degree $d$, we have

$$|\mathbf{m}X_d(i)| < |\mathbf{m}V|\,.$$

The *lex sequence* in degree $d$ consists of all the degree $d$ monomials ordered lexicographically; it is denoted by $\mathrm{Lex}_d$ or simply Lex.

**Lemma 4.4.**
*(1) $\mathrm{Lex}_d$ is a lexlike sequence.*
*(2) $X_d$ is a lexlike sequence of degree $d$ if and only if, for every $i$ we have*

$$|\mathbf{m}X_d(i)| = |\mathbf{m}\,Lex_d(i)|\,.$$

*Proof:* (1) is Macaulay's Theorem 2.18. (2) follows from (1). $\square$

Thus, lexlike sequences have minimal Hilbert function growth, as lex sequences have.

By Definition 4.3 it follows immediately that the first formulation 4.1 of Macaulay's Theorem holds for lexlike sequences:

**Theorem 4.5.** *Let $A_d$ be a monomial space in degree $d$ and $I_d$ be the space spanned by the initial segment of a lexlike sequence in degree $d$ such that $|A_d| = |I_d|$. Then $|\,\mathbf{m}I_d\,| \leq |\,\mathbf{m}A_d\,|$.*

However, it is not immediately clear that the second formulation 4.2 of Macaulay's Theorem holds for lexlike sequences. The problem is that one has to construct lexlike ideals and show that they are well defined. Here is an outline of what we do in order to extend Theorem 4.2: In each degree $d$ we have the lex sequence $\mathrm{Lex}_d$. If $L_d$ is spanned by an initial segment of $\mathrm{Lex}_d$, then $\mathbf{m}L_d$ is spanned by an initial segment of $\mathrm{Lex}_{d+1}$. This property is very easy to prove. It is very important, because it makes it possible to define lexicographic ideals. In [Me, Corollary 3.18] Mermin proved that the same property holds for lexlike sequences. This makes it possible to introduce lexlike ideals in Definition 4.9. We prove in Theorem 4.10 that Macaulay's Theorem 4.2 for lex ideals holds for lexlike ideals as well.

First, we recall a definition in [Me]: Let $X_d$ be a monomial sequence of degree $d$, and let $X_{d-1}$ be a sequence of all the monomials of $S$ of degree $d-1$. We say that $X_d$ is *above* $X_{d-1}$ if, for all $i$, there is a $j$ such that $\mathbf{m}X_{d-1}(i) = X_d(j)$. By [Me, Theorem 3.20], if $X_d$ is a monomial sequence above a lexlike sequence $X_{d-1}$, then $X_d$ is lexlike.

**Lemma 4.6.** *Let $Y$ be a lexlike sequence in degree $d$. In every degree $p$, there exists a lexlike sequence $X_p$ such that $X_d = Y$ and $X_{p+1}$ is above $X_p$ for all $p$. In particular, if a space $V_p$ is spanned by an initial segment of $X_p$, then $\mathbf{m}V_p$ is spanned by an initial segment of $X_{p+1}$.*

*Proof:* Repeatedly apply Theorem 3.21 in [Me] to get $X_p$ for $p < d$. Repeatedly apply Theorem 3.20 in [Me] to get $X_p$ for $p > d$. $\square$

**Definition 4.7**. Let $\mathbf{X}$ be a collection of lexlike sequences $X_d$ in each degree $d$, such that $X_{d+1}$ is above $X_d$ for each $d$. We call $\mathbf{X}$ a *lexlike tower.*

If we multiply a monomial sequence $X$ by a monomial $m$ by termwise multiplication, then we denote the new monomial sequence by $mX$. If $Y$ is another monomial sequence, we denote concatenation with a semicolon, so $X;Y$. Towers of monomial sequences are highly structured:

**Theorem 4.8.** *Let $\mathbf{X}$ be a lexlike tower. There exists a variable $x_i$, a lexlike tower*

$\mathbf{Y}$ *of monomials in* $S$*, and a lexlike tower* $\mathbf{Z}$ *of monomials in* $S/x_i$ *, such that*

$$\mathbf{X} = x_i\mathbf{Y}; \mathbf{Z}\,.$$

*Proof:* The variable $x_i$ is the first term of $X_1$. Each $X_d$ begins with all the degree $d$ monomials divisible by $x_i$. Writing $X_d = x_i Y_{d-1}; Z_d$ for each $d$, we have that $\mathbf{Y}$ is a lexlike tower and $\mathbf{Z}$ is a lexlike tower in $n-1$ variables. □

**Remark 4.9.** The lexicographic tower is compatible with the lexicographic order in each degree. A lexlike tower $\mathbf{X}$ induces a total ordering $<_{\mathbf{X}}$ on the monomials of $S$ which refines the partial order by degree. It is natural to ask what term orders occur this way. We show that the lexicographic order is the only one (up to reordering the variables): Suppose that $<_{\mathbf{X}}$ is a term order. Clearly $X_1$ is Lex for the corresponding order of the variables. Writing $X_2 = x_1 Y_1; Z_1$, we apply $x_1 x_i <_{\mathbf{X}} x_1 x_j$ whenever $x_i <_{\mathbf{X}} x_j$ to see that $Y_1$ is Lex and induction on $n$ to see that $Z_1$ is Lex. Thus $X_2$ is Lex. Now if $X_d = x_1 Y_{d-1}; Z_d$ is Lex, induction on $d$ and $n$ shows that $Y_d$ and $Z_{d+1}$, and hence $X_{d+1}$, are Lex as well.

In the spirit of the definition of lex ideals, we introduce lexlike ideals as follows:

**Definition 4.10**. Let $\mathbf{X}$ be a lexlike tower. We say that a $d$-vector space is an $\mathbf{X}$-space if it is spanned by an initial segment of $X_d$. We say that a homogeneous ideal $I$ is $\mathbf{X}$-lexlike if $I_d$ is an $\mathbf{X}$-space for all $d$. We say that an ideal $I$ is *lexlike* if there exists a lexlike tower $\mathbf{X}$ so that $I$ is $\mathbf{X}$-lexlike.

A lex ideal is lexlike by Lemma 4.4(1).

**Macaulay's Theorem for Lexlike Ideals 4.11.** *Let* $\mathbf{X}$ *be a lexlike tower. Let* $J$ *be a homogeneous ideal, and for each* $d$ *let* $I_d$ *be the* $\mathbf{X}$*-space spanned by the first* $|J_d|$ *monomials of* $X_d$*. Then* $I = \bigoplus I_d$ *is an* $\mathbf{X}$*-lexlike ideal and has the same Hilbert function as* $J$*.*

*Proof:* It suffices to show that $I$ is an ideal, that is, that $\mathbf{m}I_d \subset I_{d+1}$ for each degree $d$. We have $|\mathbf{m}I_d| \le |\mathbf{m}J_d| \le |J_{d+1}| = |I_{d+1}|$, and $\mathbf{m}I_d$ and $I_{d+1}$ are both spanned by initial segments of $X_{d+1}$. Since $X_{d+1}$ is above $X_d$, it follows that $\mathbf{m}I_d \subset I_{d+1}$. □

Thus, every Hilbert function is attained not only by a lex ideal (which is unique up to reordering of the variables) but also by (usually many) lexlike ideals. These distinct lexlike ideals are obtained by varying the lexlike tower $\mathbf{X}$. The following example illustrates this.

**Example 4.12**. The lexlike ideals $(ab, ac, a^3, a^2d, ad^3, b^2c)$ and $(ab, ac, ad^2, a^2d, a^4, b^4)$ have the same Hilbert function as the lex ideal $(a^2, ab, ac^2, acd, ad^3, b^4)$.

**Proposition 4.13.**

*(1) If $I$ is a lexlike ideal and $L$ is a lex ideal with the same Hilbert function, then they have the same number of minimal monomial generators in each degree.*

*(2) Among all ideals with the same Hilbert function, the lexlike ideals have the maximal number of minimal monomial generators (in each degree).*

*Proof:* (1) follows from Definition 4.1. Now, we prove (2). Macaulay's Theorem implies that among all ideals with the same Hilbert function, the lex ideal has the maximal number of minimal monomial generators (in each degree). Apply(1). □

The above theorem can be extended to all graded Betti numbers as follows:

**Theorem 4.14.**

*(1) Let $I$ be a lexlike ideal and $L$ be a lex ideal with the same Hilbert function. The graded Betti numbers of $I$ are equal to those of $L$.*

*(2) Among all ideals with the same Hilbert function, the lexlike ideals have the greatest graded Betti numbers.*

This is an extension of the following well-known result by [Bi,Hu,Pa]:

**Theorem 4.15.** [Bi,Hu,Pa] *Among all ideals with the same Hilbert function, the lex ideal has the greatest graded Betti numbers.*

*Proof of Theorem 4.14:* (2) follows from (1) and Theorem 4.15. We will prove (1).

Let $p$ be the smallest degree in which $L$ has a minimal monomial generator. For $d \geq p$, denote by $I(d)$ the ideal generated by all monomials in $I$ of degree $\leq d$. Similarly, denote by $L(d)$ the ideal generated by all monomials in $L$ of degree $\leq d$. By Lemma 4.4(2), for each $d \geq p$ the ideals $I(d)$ and $L(d)$ have the same Hilbert function. Furthermore, by Theorem 4.15 it follows that the graded Betti numbers of $S/L(d)$ are greater or equal to those of $S/I(d)$.

The following formula (cf. [Ei]) relates the graded Betti numbers $\beta_{i,j}(S/T)$ of a homogeneous ideal $T$ and its Hilbert function:

$$\sum_{j=0}^{\infty} \dim_k(S/T)_j\, t^j = \frac{\sum_{j=0}^{\infty} \sum_{i=0}^{n} (-1)^i \beta_{i,j}(S/T)\, t^j}{(1-t)^n}\,.$$

Therefore, for each $d \geq p$ we have that

$$\sum_{j=0}^{\infty} \sum_{i=0}^{n} (-1)^i \Big(\beta_{i,j}(S/I(d)) - \beta_{i,j}(S/L(d))\Big)\, t^j = 0\,. \tag{4.16}$$

By induction on $d$ we will show that the graded Betti numbers of $S/L(d)$ are equal to those of $S/I(d)$.

First, consider the case when $d = p$. By Eliahou-Kervaire's resolution [EK], it follows that $L(p)$ has a linear minimal free resolution, that is, $\beta_{i,j}(S/L(p)) = 0$ for $j \neq i+p-1$. Since the graded Betti numbers of $S/L(p)$ are greater or equal to those of $S/I(p)$, it follows that $\beta_{i,j}(S/I(p)) = 0$ for $j \neq i+p-1$. By (4.16) it follows that

$$\beta_{i,j}(S/I(p)) = \beta_{i,j}(S/L(p) \quad \text{for all } i, j\,.$$

Suppose that the claim is proved for $d$. Consider $L(d+1)$ and $I(d+1)$. For $j < i+d$, we have that

$$\beta_{i,j}(S/L(d+1)) = \beta_{i,j}(S/L(d)) = \beta_{i,j}(S/I(d))\,,$$

where the first equality follows from the Eliahou-Kervaire's resolution [EK] and the second equality holds by induction hypothesis. As $I(d+1)_q = I(d)_q$ for $q \leq d$ and since $\beta_{i,j}(S/I(d)) = 0$ for $j \geq i+d$, it follows that $\beta_{i,j}(S/I(d+1)) = \beta_{i,j}(S/I(d))$ for $j < i+d$. Therefore,

$$\begin{aligned}&\beta_{i,j}(S/L(d+1)) = \beta_{i,j}(S/I(d+1)) \text{ for } j < i+d\\ &\beta_{i,j}(S/L(d+1)) = 0 \text{ for } j > i+d, \text{ by Eliahou-Kervaire's resolution [EK].}\end{aligned}$$

Since the graded Betti numbers of $S/L(d+1)$ are greater or equal to those of $S/I(d+1)$, we conclude that

$$\begin{aligned}&\beta_{i,j}(S/I(d+1)) = \beta_{i,j}(S/L(d+1)) \text{ for } j < i+d\\ &\beta_{i,j}(S/I(d+1)) = \beta_{i,j}(S/L(d+1)) = 0 \text{ for } j > i+d\,.\end{aligned}$$

By (4.16) it follows that

$$\beta_{i,j}(S/I(d+1)) = \beta_{i,j}(S/L(d+1)) \quad \text{for all } i,j\,,$$

as desired. $\square$

**Remark 4.17.** Let $I$ be a lexlike ideal and $L$ a lex ideal with the same Hilbert function. Since their graded Betti numbers are equal, one might wonder whether the minimal free resolution $\mathbf{F}_I$ of $I$ is provided by the Eliahou-Kervaire's construction [EK]. The leading terms in the differential of $\mathbf{F}_I$ are the same as in the Eliahou-Kervaire's construction. However, the other terms could be quite different: there are examples in which the differential of $\mathbf{F}_I$ has more non-zero terms than the differential in the Eliahou-Kervaire's construction.

## 5. Multigraded Hilbert functions

In this subsection we consider the polynomial ring $S$ with a different grading, called multigrading. Such gradings are used for toric ideals. In this case, we have a multigraded Hilbert function.

Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a subset of $\mathbf{N}^c \setminus \{\mathbf{0}\}$, $A$ be the matrix with columns $a_i$, and suppose that $\operatorname{rank}(A) = c$. Consider the polynomial ring $S = k[x_1, \ldots, x_n]$ over a field $k$ generated by variables $x_1, \ldots, x_n$ in $\mathbf{N}^c$-degrees $a_1, \ldots, a_n$ respectively. We say that an ideal $J$ is $\mathcal{A}$-*multigraded* if it is homogeneous with respect to this $\mathbf{N}^c$-grading. For simplicity, we often say multigraded instead of $\mathcal{A}$-multigraded.

The prime ideal $I_{\mathcal{A}}$, that is the kernel of the homomorphism

$$\begin{aligned} \varphi:\ k[x_1, \ldots, x_n] &\to k[t_1, \ldots, t_c] \\ x_i &\mapsto \mathbf{t}^{a_i} = t_1^{a_{i1}} \ldots t_c^{a_{ic}} \end{aligned}$$

is called the *toric ideal* associated to $\mathcal{A}$. For an integer vector $\mathbf{v} = (v_1, \ldots, v_n)$ we set $\mathbf{x}^{\mathbf{v}} = x_1^{v_1} \ldots x_n^{v_n}$. Then $\varphi(\mathbf{x}^{\mathbf{v}}) = \mathbf{t}^{A\mathbf{v}}$. The *toric ring* associated to $\mathcal{A}$ is

$$S/I_{\mathcal{A}} \cong k[\mathbf{t}^{a_1}, \ldots, \mathbf{t}^{a_n}] \cong \mathbf{N}\mathcal{A}, \tag{5.1}$$

where the former isomorphism is an isomorphism of $k$-algebras and is given by $\mathbf{x}^{\mathbf{v}} \mapsto \mathbf{t}^{A\mathbf{v}}$, and the latter isomorphism is an isomorphism of monoids and is given by $\mathbf{t}^{\mathbf{a}} \mapsto \mathbf{a}$.

The ideal $I_{\mathcal{A}}$ is $\mathcal{A}$-multigraded. By (5.1), it follows that we have the multigraded Hilbert function

$$\dim_k((S/I_{\mathcal{A}})_\alpha) \;=\; \begin{cases} 1 & \text{if } \alpha \in \mathbf{N}\mathcal{A}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

There exists a minimal free resolution of $S/I_{\mathcal{A}}$ over $S$ which is $\mathbf{N}^c$-graded.

For $\alpha \in \mathbf{N}^c$, the set of all monomials in $S$ of degree $\alpha$ is called the *fiber* of $\alpha$. We introduce multilex ideals generalizing the notion of lex ideal:

**Definition 5.3.** Order the monomials in each fiber lexicographically. An $\mathcal{A}$-*multilex segment* (or multilex segment) in multidegree $\alpha$ is a vector space spanned by an initial segment of the monomials in the fiber of $\alpha$. We say that a monomial ideal $L$ is $\mathcal{A}$-*multilex* (or multilex) if for every $\alpha \in \mathbf{N}^c$, the vector space $L_\alpha$ is a multilex segment.

**Theorem 5.4.** *There exists an $\mathcal{A}$-multilex ideal $L_{\mathcal{A}}$ with the same Hilbert function as the toric ideal $I_{\mathcal{A}}$. The Betti numbers of $L_{\mathcal{A}}$ are greater or equal to those of $I_{\mathcal{A}}$.*

*Proof:* Order the monomials in each fiber lexicographically. For $\alpha \in \mathbf{N}^c$, denote by $m_\alpha$ the last monomial in the fiber of $\alpha$. Let $L_\alpha$ be the vector space spanned by all monomials in the fiber of $\alpha$ except $m_\alpha$. Set $L_{\mathcal{A}} = \oplus_\alpha L_\alpha$, where we consider $L_{\mathcal{A}}$ as a vector space. By (5.2), it follows that $L_{\mathcal{A}}$ and $I_{\mathcal{A}}$ have the same Hilbert function.

Denote by $\prec$ the lex order on monomials. We will show that $L_{\mathcal{A}}$ is the initial ideal of $I_{\mathcal{A}}$ with respect to $\prec$; in particular, $L_{\mathcal{A}}$ is an ideal. Let $m$ be a monomial in $L_\alpha$. Then $m - m_\alpha \in I_{\mathcal{A}}$ and $m \succ m_\alpha$. Hence $m$ is in the initial ideal of $I_{\mathcal{A}}$. Therefore, $L_{\mathcal{A}}$ is contained in the initial ideal. Since the multigraded Hilbert functions of $L_{\mathcal{A}}$ and $I_{\mathcal{A}}$ are the same, it follows that $L_{\mathcal{A}}$ is the initial ideal.

Clearly, $L_{\mathcal{A}}$ is multilex by construction. Since it is an initial ideal, it follows that the Betti numbers of $L_{\mathcal{A}}$ are greater or equal to those of $I_{\mathcal{A}}$. $\square$

**Example 5.5.** It should be noted that $L_{\mathcal{A}}$ depends not only on $\mathcal{A}$, but also on the choice of lexicographic order (that is, on the order of variables). For example, for the vanishing ideal $I_{\mathcal{A}} = (ad - bc,\, b^2 - ac,\, c^2 - bd)$ of the twisted cubic curve, one can get $L_{\mathcal{A}}$ to be $(ac, ad, bd)$ if $a > b > c > d$ and $(b^2, bc, bd, c^3)$ if $b > c > a > d$. These two multilex ideals have different Betti numbers.

## References


[AHH] A. Aramova, J. Herzog, and T. Hibi: Squarefree lexsegment ideals, *Math. Z.* **228** (1998), 353–378.

[Bi] A. Bigatti: Upper bounds for the Betti numbers of a given Hilbert function, *Comm. in Algebra* **21** (1993), 2317–2334.

[CL] G. Clements and B. Lindström: A generalization of a combinatorial theorem of Macaulay, *J. Combinatorial Theory* **7** (1969), 230–238.

[Ei] D. Eisenbud: *Commutative Algebra with a View Towards Algebraic Geometry*, Springer Verlag, New York 1995.

[EGH1] D. Eisenbud, M. Green, and J. Harris: Cayley-Bacharach theorems and conjectures, *Bull. Amer. Math. Soc. (N.S.)* **33** (1996), 295–324.

[EGH2] D. Eisenbud, M. Green, and J. Harris: Higher Castelnuovo theory, *Astérisque* **218** (1993), 187–202.

[EK] S. Eliahou and M. Kervaire: Minimal resolutions of some monomial ideals, *J. Algebra*, **129** (1990), 1–25.

[FR] C. Francisco and B. Richert: Lex-plus-powers ideals, preprint.

[Ga] V. Gasharov: Hilbert functions and homogeneous generic forms II, *Compositio Math.* **116** (1999), 167–172.

[Ga2] V. Gasharov: Green and Gotzmann theorems for polynomial rings with restricted powers of the variables, *J. Pure Appl. Algebra* **130** (1998), 113–118.

[GHP] V. Gasharov, T. Hibi, I. Peeva: Resolutions of **a**-stable ideals, *J. Algebra* **254** (2002), 375–394.

[Gr] M. Green: Generic initial ideals, in *Six lectures on commutative algebra*, Birkhäuser, Progress in Mathematics **166**, (1998), 119–185.

[Gr2] M. Green: Restrictions of linear series to hyperplanes, and some results of Macaulay and Gotzmann, in *Algebraic curves and projective geometry*, Lecture Notes in Math. **1389** (1989), 76–86.

[HP] J.Herzog and D. Popescu: Hilbert functions and generic forms, *Compositio Math.* **113** (1998), 1–22.

[Hu] H. Hulett: Maximum Betti numbers of homogeneous ideals with a given Hilbert function, *Comm. in Algebra* **21** (1993), 2335–2350.

[Ma] F. Macaulay: Some properties of enumeration in the theory of modular systems, *Proc. London Math. Soc.* **26** (1927), 531–555.

[MP] J. Mermin: Lexlike sequences, in preparation.

[MP] J. Mermin and I. Peeva: Lexifying Ideals, submitted.

[Pa] K. Pardue: Deformation classes of graded modules and maximal Betti numbers, *Ill. J. Math.* **40** (1996), 564–585.

[Sh] D. A. Shakin: Piecewise lexsegment ideals, *Mat. Sbornik* **194** (2003), 1701-1724