Part II. Team Round

1. A **primitive Pythagorean triple** is an ordered triple of positive integers (a, b, c) satisfying the equation $a^2 + b^2 = c^2$, and such that $\gcd(a, b, c) = 1$.

   (a) Find all primitive Pythagorean triples which are also arithmetic sequences, and prove that your list is complete.
   (b) Find all primitive Pythagorean triples with $a < b < c < 30$, and prove that your list is complete.
   (c) Find a primitive Pythagorean triple (a, b, c) with $2013 < a < b < c$.

   Solution: For part (a), assume that the arithmetic sequence (a, b, c) is a primitive Pythagorean triple, and write $a = b - d$ and $c = b + d$. Then we have

$$\begin{aligned}
(b-d)^2 + b^2 &= (b+d)^2 \\
b^2 - 2bd + d^2 + b^2 &= b^2 + 2bd + d^2 \\
b^2 - 2bd &= 2bd \\
b^2 - 4bd &= 0 \\
b(b - 4d) &= 0
\end{aligned}$$

   Since b is positive, we know $b \neq 0$. Thus $b = 4d$, so our triple can be rewritten in the form $(3d, 4d, 5d)$. Since the greatest common divisor must be 1, we conclude that $d = 1$, so our triple must have been $(3, 4, 5)$. Hence $\boxed{(3, 4, 5)}$ is the only primitive Pythagorean triple that is also an arithmetic sequence.

   For part (b), we will use the following famous result:

   > **Theorem**: Every primitive Pythagorean triple has the form $(m^2 - n^2, 2mn, m^2 + n^2)$ for some positive integers m and n.
   >
   > *Proof:* Let (a, b, c) be a primitive Pythagorean triple. First we claim that no prime number divides more than one of a, b, or c. Indeed, if p divided both a and b, then it would also divide $c^2 = a^2 + b^2$, so it would have to divide c as well, and consequently we would have $\gcd(a, b, c) \neq 1$. (The other possibilities are identical.) In particular, it follows that 2 does not divide both a and b, so at least one of these is odd. Now assume without loss of generality that a is odd. Observe that a^2 has a remainder of 1 upon division by four, and that b^2 and c^2 have remainders of either 0 or 1 upon division by four. The only way this can work is if b is even and c is odd.
   >
   > Now set $M = \frac{c+a}{2}$ and $N = \frac{c-a}{2}$, so that $a = M - N$ and $c = M + N$. We claim that M and N are perfect squares. Observe that $MN = \frac{b^2}{4}$ is a perfect square. Thus, if $\gcd(M, N) = 1$, it follows that both are squares. On the other hand, if $\gcd(M, N) = p \neq 1$, then p divides both $c = M + N$ and $a = M - N$, which we proved above is impossible. Taking $m = \sqrt{M}$ and $n = \sqrt{N}$ yields $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$ as desired.

Using this theorem, it suffices to check all pairs of positive numbers $1 \le m < n \le 5$ (because $6^2 > 30$):

m	n	(a, b, c)
1	2	$(3, 4, 5)$
1	3	$(6, 8, 10)$
1	4	$(8, 15, 17)$
1	5	$(10, 24, 26)$
2	3	$(5, 12, 13)$
2	4	$(12, 16, 20)$
2	5	$(20, 21, 29)$
3	4	$(7, 24, 25)$
3	5	$(16, 30, 34)$
4	5	$(9, 40, 41)$

The primitive triples on this list are

$$\boxed{(3, 4, 5),\ (8, 15, 17),\ (5, 12, 13),\ (20, 21, 29),\ \text{and } (7, 24, 25)}$$

(and $(9, 40, 41)$, which has entries greater than 30).

For (c), we could try large numbers for m and n until we found an appropriate primitive Pythagorean triple, but that might prove difficult. Another possibility is to recognize that, for every odd n, the triple $(n, \frac{n^2-1}{2}, \frac{n^2+1}{2})$ is a primitive Pythagorean triple. Thus $\boxed{(2015, 2030112, 2030113)}$ works, as does $\boxed{(10001, 50010000, 50010001)}$ (for easier arithmetic). Similarly, if n is divisible by 4, the triple $(n, \frac{n^2}{4} - 1, \frac{n^2}{4} + 1)$ is a primitive Pythagorean triple, so $\boxed{(2016, 1016063, 1016065)}$ and $\boxed{(10000, 24999999, 25000001)}$ both work.

2. Call a matrix *even* if its entries are either zero or one, and the sum of each row and column is even. For instance, the first matrix below is even, while the second is not.

$$\text{Even: } \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{Not even: } \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

(a) How many 3×3 even matrices are there?
(b) How many 4×4 even matrices are there?
(c) How many 2013×2013 even matrices are there?

Solution: For an integer n, let the **parity** of n be 0 if n is even and 1 if n is odd. (This is the same as the value of n modulo two.)

For part (a), let $M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. We claim that, after choosing values for a_{11}, a_{12}, a_{21}, and a_{22}, there is exactly one way to assign values to the other five entries to produce an even matrix. Indeed, a_{13} must be the parity of $r_1 = a_{11} + a_{12}$ and a_{23} must be the parity of $r_2 = a_{21} + a_{22}$ in order for the first and second row-sums to be even. Likewise, a_{31} and a_{32} must be the parities of $c_1 = a_{11} + a_{21}$ and $c_2 = a_{12} + a_{22}$, respectively, for the column-sums to be even. Finally, a_{33} must equal the parity of $c_1 + c_2$ for the final row-sum to be even, and must also equal the parity of $r_1 + r_2$ for the final column-sum to be even. Since $c_1 + c_2 = a_{11} + a_{12} + a_{21} + a_{22} = r_1 + r_2$, we can satisfy both conditions.

There are $2 \times 2 \times 2 \times 2 = 2^4 = 16$ ways to choose values for a_{11}, a_{12}, a_{21}, and a_{22}, and each choice yields exactly one even matrix, so there are $\boxed{16}$ 3×3 even matrices.

For parts (b) and (c), we generalize the argument about to compute the number of $n \times n$ even matrices. Let a_{ij} represent the entry in the i^{th} row and j^{th} column of the $n \times n$ matrix M. The condition that M is even is exactly equivalent to its entries all being either zero or one and the following three conditions being satisfied:

- a_{in} is the parity of $r_i = a_{i1} + \cdots + a_{i,n-1}$ for all i between 1 and $n - 1$.
- a_{nj} is the parity of $c_j = a_{1j} + \cdots + a_{n-1,j}$ for all j between 1 and $n - 1$.
- a_{nn} is the parity of $r_1 + \cdots + r_{n-1}$ and a_{nn} is the parity of $c_1 + \cdots + c_{n-1}$.

Since $r_1 + \cdots + r_{n-1}$ and $c_1 + \cdots + c_{n-1}$ both expand to the sum $\sum_{i,j \le n-1} a_{ij}$ of all the entries not in the bottom row or rightmost column, it is always possible to satisfy the right-most condition. Thus every $(n-1) \times (n-1)$ matrix with entries in $\{0, 1\}$ is the upper-left block of exactly one $n \times n$ even matrix. There are $2^{(n-1)^2}$ ways to choose such an upper-left block, so there are exactly $2^{(n-1)^2}$ even $n \times n$ matrices. In particular, the answer to part (b) is $2^9 = \boxed{512}$, and the answer to part (c) is $\boxed{2^{2012^2}}$.

3. Alice writes the numbers $1, 2, \ldots, 2013$ on a chalkboard. Then she chooses two numbers a and b, erases them, and replaces them with the new number $ab + a + b$. She repeats this process 2012 times until only one number remains. Determine this final number, and prove that the order in which Alice chose numbers to erase doesn't matter.

   Solution: Meanwhile, across the room, Alice's friend Alison, working on a different chalkboard, writes down (or erases) $x + 1$ whenever Alice writes (or erases) x. Thus Alison starts with the numbers $2, 3, \ldots, 2014$, and if, for example, Alice erases the numbers 2 and 4 and replaces them with 14, Alison will replace 3 and 5 with 15.

   When Alice replaces the numbers a and b with $ab + a + b$, Alison replaces $a + 1$ and $b + 1$ with $(ab + a + b) + 1$. Since $ab + a + b + 1 = (a + 1)(b + 1)$, Alison is simply performing multiplication.

   Thus at the end of the process, Alison is left with the product of her starting numbers, $(2)(3)\ldots(2014) = 2014!$. Alice's final number is one less than Alison's, $\boxed{2014! - 1}$.

   Because multiplication is associative, Alison's final number doesn't depend on Alice's choices. Consequently, Alice's final number doesn't depend on those choices either.

4. Prove that any rectangular prism with volume 125 cubic units and surface area of 150 square units must be a cube.

   Solution: We use the **Arithmetic Mean - Geometric Mean (AM-GM) inequality**:

   > Let $x_1, \ldots, x_n$ be positive real numbers. Then the arithmetic mean of these numbers, $\dfrac{x_1 + \cdots + x_n}{n}$, is greater than or equal to their geometric mean, $\sqrt[n]{x_1 \ldots x_n}$. Furthermore, the two means are equal if and only if $x_1 = x_2 = \cdots = x_n$.

   We first prove that any *right* rectangular prism with the given volume and surface area is a cube. A right rectangular prism with length ℓ, width w, and height h has volume ℓwh and surface area $2(\ell w + \ell h + wh)$. Thus, if we set AM and GM equal to the arithmetic and geometric means, respectively, of ℓw, ℓh, and wh, we get $\mathrm{AM} = \frac{SA}{6}$ and $\mathrm{GM} = V^{\frac{2}{3}}$. In our case, we have $\mathrm{AM} = \mathrm{GM} = 25$, from which we conclude that $\ell w = \ell h = wh = 25$, so $\ell = w = h = 5$ and our prism is a cube.

   Now we prove that any rectangular prism with the given volume and surface area must be right. Suppose that some oblique rectangular prism has surface area 150. Then we may stand it so that a rectangular face is horizontal. Then the prism has volume ℓwh and surface area $\ell w + \ell s + wt$, where s and t are the slant-heights of the oblique faces adjacent to ℓ and w, respectively. We know that s and t are both greater than or equal to h; since our prism is oblique, at least one of them is strictly greater. Thus, applying the AM-GM inequality to ℓw, ℓh, and wh again, we have:

$$\begin{aligned} 2(\ell w + \ell h + wh) &\lneqq 150 \\ \mathrm{AM} &\lneqq 25 \\ \mathrm{GM} &\lneqq 25 \\ \ell wh &\lneqq 125. \end{aligned}$$

   Thus any rectangular prism with volume 125 and surface area 150 must be right and so, by the argument above, must be a cube.

5. A carnival game works as follows: An integer x is chosen at random between 1 and 2013 (inclusive). The player may choose between two moves: For a cost of two tokens, he may choose a number y and learn the answer to the question "Is x less than y"; if the answer is "no", he is refunded one of the two tokens. Alternatively, for a cost of five tokens, he may choose a number y and learn the answer to the question "Does x equal y?"; if the answer is "yes", he wins a fabulous prize.
(a) Prove that it is possible to guarantee a fabulous prize if you start with 2017 tokens.
(b) Prove that it is possible to guarantee a fabulous prize if you start with 27 tokens.
(c) What is the smallest number of tokens necessary to guarantee a fabulous prize?

Solution: Any solution to part (c) will contain a solution to parts (a) and (b), so we restrict our attention to part (c).

We begin by generalizing the problem. Let $G(m, n)$ be the game described in the problem statement, but with the winning number chosen between m and n instead of between 1 and 2013. Observe that $G(m, n)$ is equivalent to $G(1, n - m + 1)$ (just subtract $(m - 1)$ from every number), and so a strategy for winning $G(1, n - m + 1)$ is equivalent to a strategy for $G(m, n)$. We will find the smallest number of tokens necessary to guarantee a win at $G(1, n)$.

A strategy to guarantee a win in $G(1, n)$ using k tokens must take one of the following two forms:

Strategy One: Choose a number y, and ask if $x = y$. Then either claim the prize, or win $G(1, n)$ using $k - 5$ tokens and the knowledge that $x \neq y$.

Strategy Two: Choose a number y, and ask if $x < y$. Then, depending on the answer, either win $G(y, n)$ using $k - 1$ tokens or win $G(1, y - 1)$ using $k - 2$ tokens.

If $n = 1$, then strategy one is clearly correct. However, if $n \geq 2$ (so that it is possible to guess wrong), strategy one makes a suboptimal guarantee: We can guarantee a win using $k - 1$ tokens with the following strategy: Choose the same y, and ask if $x < y$ and if $x < y + 1$. (This costs at most four tokens.) If we can then conclude that $x = y$, claim the prize, having spent only nine tokens. Otherwise, win the game using the knowledge that $x \neq y$ and $k - 5$ tokens as we would have in strategy one.

Thus we want to understand the efficiency of strategy two. Let $E(k)$ be the largest n such that we can guarantee a win to $G(1, n)$ using k tokens. Clearly, $E(5) = E(6) = 1$. Otherwise, if $n \leq E(k)$, then, for an appropriate choice of y, it must be the case that $G(1, y-1)$ is winnable using $k-2$ tokens (so $y-1 \leq E(k-2)$) and $G(y, n)$ is winnable using $k-1$ tokens (so $n-y+1 \leq E(k-1)$). This proves that $E(k) \leq E(k-2)+E(k-1)$. On the other hand, if $n = E(k - 2) + E(k - 1)$, we can win with k tokens by choosing $y = E(k - 2) + 1 = n - E(k - 1) + 1$.

Thus $E(k) = E(k-1) + E(k-2)$ and $E(5) = E(6) = 1$, so the numbers $E(k)$ are simply the Fibonacci numbers offset by five. We compute $E(7) = 2$, $E(8) = 3$, $E(9) = 5$, and so on. Eventually, we find $E(21) = 1597$, $E(22) = 2584$.

Thus it is possible to guarantee a win to $G(1, 2013)$ using $\boxed{22}$ tokens.

For example, if $x = 270$, our strategy will take the following sequence of y:

Game	Size	y	Information	Tokens
$G(1, 2013)$	2013	988	$x \in [1, 987]$	20
$G(1, 987)$	987	378	$x \in [1, 377]$	18
$G(1, 377)$	377	145	$x \in [145, 377]$	17
$G(145, 377)$	233	234	$x \in [234, 377]$	16
$G(234, 377)$	144	289	$x \in [234, 288]$	14
$G(234, 288)$	55	255	$x \in [255, 288]$	13
$G(255, 288)$	34	268	$x \in [268, 288]$	12
$G(268, 288)$	21	276	$x \in [268, 275]$	10
$G(268, 275)$	8	271	$x \in [268, 270]$	8
$G(268, 270)$	3	269	$x \in [269, 270]$	7
$G(269, 270)$	2	270	$x = 270$	6

We claim our prize with five of the remaining tokens.