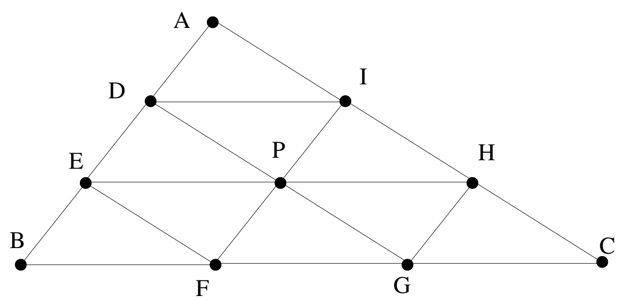
1. Let ABC be a triangle with area 9. Prove that it is possible to subdivide ABC into nine smaller triangles each with area 1. Then prove that there is a point P on the interior of ABC such that every line through P divides ABC into two regions each with area between 4 and 5.

   Solution:



   Let D, E, F, G, H, and I trisect sides AB, BC, and CA as in the figure. Observe that ABC, DBG, and EBF share side-angle-side similarity; thus EF and DG are parallel to AC. Similarly DI and EH are parallel to BC, and GH and FI are parallel to AB.

   Let P be the intersection of EH and DG. We claim that P is the midpoint of both EH and DG. Observe that DEP is similar to ABC (because of the parallel sides), and $DE = \frac{1}{3}AB$. Thus $EP = \frac{1}{3}BC = \frac{1}{2}EH$, so P is the midpoint of EH. Similarly, it is the midpoint of DG.

   The same argument shows that EH and FI intersect in the midpoint of EH, so we conclude that EH, DG, and FI all meet at P.

   Now triangles ADI, DEP, IPH, EBF, PFG, HGC, PID, FPE, and GHP are all similar to ABC with length ratio $\frac{1}{3}$. Thus their areas are all equal to $\frac{9}{3^2} = 1$.

   Now suppose ℓ is any line through P. Suppose ℓ passes through triangles PID and PFG (the other cases will be similar). Let J and K be the intersections of ℓ with DI and FG, respectively. Then PIJ and PFK are congruent by side-angle-side. Let X be the part of ABC to the left of ℓ. Then we have

$$\begin{aligned}
X &= EBF \cup PFE \cup DEP \cup PJD \cup PFK \cup (\text{part of } ADI) \\
\text{area}(X) &\geq \text{area}(EBF) + \text{area}(PFE) + \text{area}(DEP) + \text{area}(PJD) + \text{area}(PFK) \\
&\geq 1 + 1 + 1 + (\text{area}(PJD) + \text{area}(PIJ)) \\
&\geq 1 + 1 + 1 + \text{area}(PID) \\
&\geq 4.
\end{aligned}$$

   Similarly, the area to the right of ℓ is at least 4. Since these add to 9, they must both be between four and five.

2. Factor the polynomial $p(x) = x^8 + x^4 + 1$ as a product of three nontrivial polynomials with integer coefficients. Describe the roots of p.

   Solution: Observe that $p(x) = 1 + x^4 + x^8$ is a geometric series with common ratio x^4. Thus $p(x) = \frac{x^{12}-1}{x^4-1}$.

   We factor the numerator

$$\begin{aligned} x^{12} - 1 &= (x^6 + 1)(x^6 - 1) \\ &= (x^2 + 1)(x^4 - x^2 + 1)(x^3 + 1)(x^3 - 1) \\ &= (x^2 + 1)(x^4 - x^2 + 1)(x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1). \end{aligned}$$

   Similarly, we factor the denominator,

$$\begin{aligned} x^4 - 1 &= (x^2 + 1)(x^2 - 1) \\ &= (x^2 + 1)(x + 1)(x - 1). \end{aligned}$$

   Cancelling, we have $p(x) = (x^4 - x^2 + 1)(x^2 - x + 1)(x^2 + x + 1)$.

   The roots of p are the roots of $x^{12} - 1$ which are not roots of $x^4 - 1$, that is, the complex numbers other than ± 1, $\pm i$ which satisfy $x^{12} = 1$. They are

$$\{e^{\frac{\pi i}{6}}, e^{\frac{\pi i}{3}}, e^{\frac{2\pi i}{3}}, e^{\frac{5\pi i}{6}}, e^{\frac{7\pi i}{6}}, e^{\frac{4\pi i}{3}}, e^{\frac{5\pi i}{3}}, e^{\frac{11\pi i}{6}}\}.$$

3. Suppose that a polygon P is invariant under the rotation about a given point c by an angle of $48°$. (This means the polygon obtained after the rotation coincides with P. For example, a square is invariant under a rotation about its center by $90°$, by not by $45°$.)

   (a.) Is P necessarily invariant under a rotation about c by $90°$?

   (b.) Is P necessarily invariant under a rotation about c by $72°$?

   (c.) Is P necessarily invariant under a rotation about c by $120°$?

   Solution: Observe that P is invariant under a rotation about c by $360°$, since this rotation fixes the entire plane. Also observe that if P is invariant under rotations about c by angles α and β, then it is invariant under rotations about c by $A\alpha + B\beta$, for any integers A and B.

   Since $24° = 360° - 7 \cdot 48°$, it follows that P is invariant under a rotation about c by $24°$ and by any multiple of $24°$. Thus in particular it is invariant under a rotation about c by $72°$, and under a rotation about c by $120°$.

   However, 90 is not a multiple of 24, so part **(a)** is still open. Observe that a regular 15-gon is invariant under rotation about its center by $\frac{360°}{15} = 24°$ and hence under a rotation about its center by $48°$. However, the 15-gon is not invariant under rotation by $90°$ about its center. Thus, P is not necessarily invariant under a rotation about c by $90°$.

4. The Fibonacci sequence is defined by $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all positive integers n. Prove that $F_{n+10} - F_n$ is divisible by 11 for all positive n.

Solution: We have:

$$\begin{aligned}
F_{n+10} &= F_{n+9} + F_{n+8} & &= F_{n+9} + F_{n+8} \\
&= F_{n+8} + F_{n+7} + F_{n+8} & &= 2F_{n+8} + F_{n+7} \\
&= 2F_{n+7} + 2F_{n+6} + F_{n+7} & &= 3F_{n+7} + 2F_{n+6} \\
&= 3F_{n+6} + 3F_{n+5} + 2F_{n+6} & &= 5F_{n+6} + 3F_{n+5} \\
&= \ldots & &= 55F_{n+1} + 34F_n.
\end{aligned}$$

(One can brute-force this, or prove inductively that $F_aF_b + F_{a-1}F_{b-1} = F_{a+b}$.)

Thus $F_{n+10} - F_n = 55F_{n+1} + 33F_{n+1} = 11(5F_{n+1} + 3F_n)$ is divisible by 11.

Alternative Solution: Observe that $F_{10} = 89$ and $F_{11} = 144$. Thus $F_{10} - F_0 = 88$ and $F_{11} - F_1 = 143$. In particular, these are both divisible by 11.

Inductively, suppose $F_{k+9} + F_{k-1} = 11a$ and $F_{k+10} + F_k = 11b$ are both divisible by 11. Then

$$\begin{aligned}
F_{k+11} - F_{k+1} &= F_{k+10} + F_{k+9} - (F_k + F_{k-1}) \\
&= (F_{k+10} - F_k) + (F_{k+9} - F_{k-1}) \\
&= 11(a + b)
\end{aligned}$$

is also divisible by 11. Thus $F_{n+10} - F_n$ is divisible by 11 for all n.

5. Find all real numbers a such that the polynomial $x^{2011} - ax^{2010} + ax - 1$ is divisible by $(x-1)^2$.

Solution: Set $y = x - 1$; then we are looking for the set of all a such that

$$p(y) = (y+1)^{2011} - a(y+1)^{2010} + a(y+1) - 1$$

is divisible by y^2.

Expanding with the binomial theorem, we have

$$\begin{aligned} p(y) = 1 &+ 2011y + \text{terms divisible by } y^2 \\ &- a\left(1 + 2010y + \text{terms divisible by } y^2\right) \\ &+ ay + a - 1, \end{aligned}$$

that is,

$$p(y) = (1 - a + a - 1) + (2011 - 2010a + a)y + \text{terms divisible by } y^2.$$

This is divisible by y^2 if and only if $2011 - 2010a + a = 0$.

The only solution is $a = \frac{2011}{2009}$.

Alternate solution: Let $f(x) = x^{2011} - ax^{2010} + ax - 1$. We have

$$\begin{aligned} f(x) &= (x^{2011} - 1) - ax(x^{2009} - 1) \\ &= (x-1)(x^{2010} + (1-a)x^{2009} + (1-a)x^{2008} + \cdots + (1-a)x + 1). \end{aligned}$$

Set $g(x) = x^{2010} + (1-a)x^{2009} + (1-a)x^{2008} + \cdots + (1-a)x + 1$. Then $f(x)$ is divisible by $(x-1)^2$ if and only if $g(x)$ is divisible by $(x-1)$. By the Remainder Theorem, $g(x)$ is divisible by $(x-1)$ if and only if $g(1) = 0$. We compute $g(1) = 1 + 2009(1-a) + 1 = 2011 - 2009a$. This is equal to zero if and only if $a = \frac{2011}{2009}$.

Solution using calculus: In general, a polynomial $p(x)$ is divisible by $(x-c)^2$ if and only both $p(x)$ and $p'(x)$ are divisible by $(x-c)$. Here, $p'(x) = 2011x^{2010} - 2010ax^{2009} + a$. By the remainder test, $p(x)$ is always divisible by $x-1$, and $p'(x)$ is divisible by $(x-1)$ if and only if $2011 - 2010a + a = 0$, i.e., $a = \frac{2011}{2009}$.