Part I. Individual Round

1. What is the largest possible number of intersections between a circle and a parabola?

   A. 2

   B. 4

   C. 6

   D. 8

   E. 10

2. Twelve cowboys sit in a circle around a bonfire. Each observes that his age (viewed as an integer) is the average of the ages of his left and right neighbors. Which of the following could be the sum of their ages?

   A. 224

   B. 225

   C. 226

   D. 227

   E. 228

3. How many distinct solutions does the equation $|||x-1|-1|-1| = 2013$ have?

   A. None

   B. One

   C. Two

   D. Four

   E. Eight

4. Roquentin has six books, two of which are by Sartre. How many ways can Roquentin arrange his books on a shelf, assuming the two books by Sartre must always be next to one another?

   A. 48

   B. 120

   C. 240

   D. 360

   E. 720

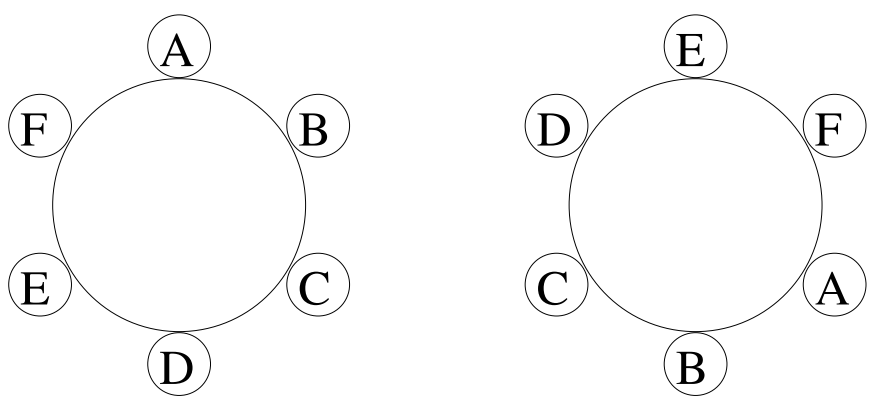
5. What is the remainder when $x^3 - 2x^2 - 1$ is divided by $x + 2$?

   A. -15
   B. 15
   C. -1
   D. -17
   E. 17

6. Pieces in the shape of an equilateral triangle of side length 10 cm are used to tile a mosaic in the shape of an equilateral triangle with side length 2 meters. How many such pieces are needed?

   A. 20
   B. 40
   C. 200
   D. 400
   E. 720

7. Three married couples sit around a round table. In how many ways can they be seated if each person must be directly opposite his/her spouse? (We count seating arrangements *up to rotation*. That is, if one seating arrangement can be rotated into another, we consider them to be the same seating arrangement. For example, the two arrangements below are the same.)



   A. 3
   B. 5
   C. 6
   D. 8
   E. 12

8. Let S be the set of points that are exactly twice as far from $(2,0)$ as they are from $(1,3)$. Find the equation of S.

   A. $9x^2 + y^2 - 6xy + 11x - 33y + 18 = 0$

   B. $x^2 + 9y^2 - 6xy + 39x - 107y + 318 = 0$

   C. $3x - 9y + 14 = 0$

   D. $3x^2 + 3y^2 - 4x - 24y + 36 = 0.$

   E. $3x + y - 6 = 0$

9. Simplify the expression
$$\log_2 6 + \log_4 18 - \log_2 9.$$

   A. $\frac{3}{2}$

   B. 0

   C. $\log_2 15$

   D. $\log_4 12$

   E. $\log_2 6$

10. What is the sum of all odd two-digit numbers? (One-digit numbers are *not* two-digit numbers.)

   A. 2475

   B. 2500

   C. 2525

   D. 2550

   E. 2575

11. Find the period of the function $y = \cos(6x) + \cos(9x)$.

   A. $\dfrac{\pi}{3}$

   B. $\dfrac{\pi}{2}$

   C. $\dfrac{2\pi}{3}$

   D. 2π

   E. 36π

12. Find the distance between the circles $(x-10)^2+(y+4)^2 = 169$ and $(x+5)^2+(y-4)^2 = 1$.

A. 3

B. 4

C. 5

D. 6

E. 7

13. What is the sum of the positive divisors of 2013? (1 and 2013 are divisors of 2013.)

A. 2014

B. 2688

C. 2760

D. 2793

E. 2976

14. Find the distance from the point $(1, 2)$ to the line $3x + 4y = 5$.

A. $\frac{\sqrt{6}}{5}$

B. $\frac{6}{5}$

C. $\frac{11}{5}$

D. 6

E. $\frac{6\sqrt{5}}{5}$

15. How many pairs of positive integers (x, y) satisfy $20x + 13y = 2013$?

A. 4

B. 8

C. 13

D. 33

E. 100

16. If $\sin a + 8\sin b + 15\cos b = 18$, then what is the absolute value of $\cos a$?

A. 0

B. $\frac{2}{3}$

C. $\sqrt{\frac{8}{15}}$

D. $\sqrt{\frac{15}{18}}$

E. 1

17. Find the maximum possible value of $x + y$ on the region in the first quadrant defined by the inequalities $x + 2y \leq 6$ and $4y + 5x \leq 20$.

A. 4

B. $\frac{49}{12}$

C. $\frac{25}{6}$

D. $\frac{17}{4}$

E. $\frac{13}{3}$

18. What is the largest root of the equation

$$(1-x)^2 + (2-x)^2 + \cdots + (2013-x)^2 = 671 \cdot 1007 \cdot 4027?$$

A. $\frac{4027 + \sqrt{1007}}{671}$

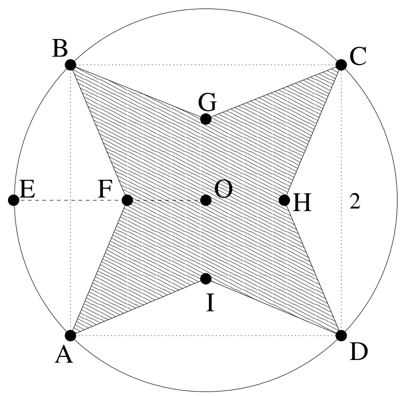
B. This equation has no real roots.

C. 2014

D. 0

E. $\frac{-671 + \sqrt{1007 \cdot 4027}}{2}$

19. In the figure below, square $ABCD$ has sides of length 2 and is inscribed in circle O. E, F, and O are collinear, and EF and AB are perpendicular bisectors. Assuming star $AFBGCHDI$ has $90°$ rotational symmetry, what is its area?



A. $1 + 2\sqrt{2}$

B. $2\sqrt{2}$

C. $2 + \sqrt{2}$

D. $8 - 4\sqrt{2}$

E. $2\sqrt{2} - 2$

20. If $x^2 - x + 1 = 0$, then what is $x^{2013} - x^{671} + x^{11}$?

A. 0

B. $2013 - 1353\sqrt{3}$

C. $\dfrac{1353\sqrt{3}}{2}$

D. 3

E. -1

21. A regular dodecagon is inscribed in a circle of radius 1. What is the area outside the dodecagon but inside the circle? (A dodecagon has twelve sides.)

   A. $\dfrac{\pi}{12}$

   B. $\pi - \dfrac{5\sqrt{6}}{4}$

   C. $\pi - \dfrac{22}{7}$

   D. $\pi - \dfrac{3\sqrt{3}}{2}$

   E. $\pi - 3$

22. What is the largest integer n such that 2^n divides the product

$$P = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot 2013?$$

   A. 2000

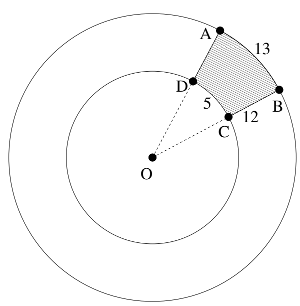
   B. 2001

   C. 2002

   D. 2003

   E. 2004

23. In the figure below, the two circles are concentric with shared center O. A, D, and O are collinear, as are B, C, and O. Given that $\overset{\frown}{\mathrm{AB}} = 13$, $\overline{BC} = 12$, and $\overset{\frown}{\mathrm{CD}} = 5$, find the area of the shaded region.



   A. 30

   B. 30π

   C. 78

   D. $\dfrac{390}{\pi}$

   E. 108

24. Bob, Mack, and Mike share ownership of a ping-pong table. They draw straws to decide who plays the first game, then after each game the loser sits out while the winner plays against the previous sit-out. When they are done, Mike has played 15 games and Bob has played 7. How many games were played in total?

A. 8

B. 15

C. 17

D. 19

E. 22

25. Let $ABCD$ be a quadrilateral inscribed in a circle of radius 25, such that $AB = 14$, $BC = 48$, and $AC \perp BD$. Find BD.

A. $\frac{1415}{48}$

B. $\frac{977}{35}$

C. $\frac{672}{25}$

D. $\frac{387}{14}$

E. $\frac{121}{9}$

Part II. Team Round

1. A **primitive Pythagorean triple** is an ordered triple of positive integers (a, b, c) satisfying the equation $a^2 + b^2 = c^2$, and such that $\gcd(a, b, c) = 1$.

   (a) Find all primitive Pythagorean triples which are also arithmetic sequences, and prove that your list is complete.
   (b) Find all primitive Pythagorean triples with $a < b < c < 30$, and prove that your list is complete.
   (c) Find a primitive Pythagorean triple (a, b, c) with $2013 < a < b < c$.

2. Call a matrix *even* if its entries are either zero or one, and the sum of each row and column is even. For instance, the first matrix below is even, while the second is not.

$$\text{Even: } \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{Not even: } \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

(a) How many 3×3 even matrices are there?
(b) How many 4×4 even matrices are there?
(c) How many 2013×2013 even matrices are there?

3. Alice writes the numbers $1, 2, \ldots, 2013$ on a chalkboard. Then she chooses two numbers a and b, erases them, and replaces them with the new number $ab + a + b$. She repeats this process 2012 times until only one number remains. Determine this final number, and prove that the order in which Alice chose numbers to erase doesn't matter.

4. Prove that any rectangular prism with volume 125 cube units and surface area of 150 square units must be a cube.

5. A carnival game works as follows: An integer x is chosen at random between 1 and 2013 (inclusive). The player may choose between two moves: For a cost of two tokens, he may choose a number y and learn the answer to the question "Is x less than y"; if the answer is "no", he is refunded one of the two tokens. Alternatively, for a cost of five tokens, he may choose a number y and learn the answer to the question "Does x equal y?"; if the answer is "yes", he wins a fabulous prize.
(a) Prove that it is possible to guarantee a fabulous prize if you start with 2017 tokens.
(b) Prove that it is possible to guarantee a fabulous prize if you start with 27 tokens.
(c) What is the smallest number of tokens necessary to guarantee a fabulous prize?