Part I. Individual Round

1. The largest possible number of Sundays in a year is

   A. 51

   B. 52

   C. 53

   D. 61

   E. 365

2. Let $i = \sqrt{-1}$. Compute $(20 + 11i)^2$.

   A. 521

   B. 279

   C. $400 + 121i$

   D. $521 + 440i$

   E. $279 + 440i$

3. In triangle ABC, denote the lengths of edges opposite to $\angle A$, $\angle B$, $\angle C$ by a, b, c, respectively. If $\angle A : \angle B : \angle C = 1 : 2 : 3$, then $a : b : c$ is

   A. $1 : 2 : 3$

   B. $1 : \sqrt{3} : 2$

   C. $2 : \sqrt{3} : 1$

   D. $\sqrt{3} : \sqrt{2} : 1$

   E. $1 : \sqrt{2} : \sqrt{3}$

4. Two fair six-sided dice are rolled. What is the probability that the results sum to two?

   A. $\frac{1}{36}$

   B. $\frac{1}{12}$

   C. $\frac{1}{11}$

   D. $\frac{1}{6}$

   E. $\frac{1}{3}$

5. The system of equations $x^2 + y^2 = 1,\ y - x^3 = 0$ has

A. No real solution

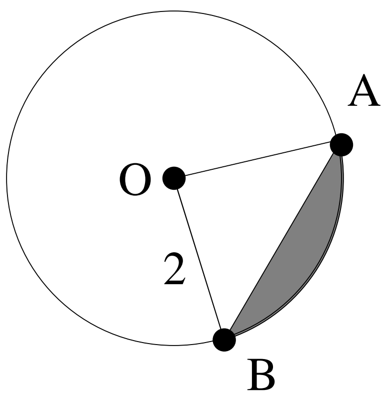
B. 1 real solution

C. 2 real solutions

D. 3 real solutions

E. 4 real solutions

6. In the figure below (not drawn to scale), A and B are points on circle O, and ABO is an equilateral triangle. What is the area of the shaded region?



A. $\frac{6-2\pi}{3}$

B. $\frac{3\sqrt{3}-2\pi}{3}$

C. $\frac{2\pi-6}{3}$

D. $\frac{2\pi}{3}$

E. $\frac{2\pi-3\sqrt{3}}{3}$

7. Which of the integers below (all written in some base other than 10) is a perfect square?

A. 20_8

B. 20_9

C. 20_{11}

D. 20_{12}

E. 20_{13}

8. Which of the expressions below is equal to $\arcsin(x) + \arccos(x)$ for all x between -1 and 1?

   A. x^2

   B. $\frac{\pi}{2}$

   C. $2x$

   D. x

   E. 0

9. A triangle with integer edge lengths has an edge of length 3. If 3 is not the shortest edge length, how many sets of edge lengths are possible for this triangle?

   A. 2

   B. 3

   C. 4

   D. 5

   E. 6

10. On a 20-question multiple-choice exam, students receive $+8$ points for each correct answer, -5 for each incorrect answer, and 0 points for each question left blank. A student earns a score of 13 on this exam. How many questions did he attempt?

    A. 6

    B. 7

    C. 10

    D. 12

    E. 13

11. What is the largest number of equal squares one can make out of 17 identical toothpicks, if every side of a square consists of one toothpick?

    A. 4

    B. 5

    C. 6

    D. 7

    E. 8

12. A cube has 8 vertices. How many acute triangles can be formed using these vertices?

A. 0

B. 6

C. 8

D. 24

E. 56

13. If $\sin\theta + \cos\theta = 1$, then what is $(\sin\theta)^{2011} + (\cos\theta)^{2011}$?

A. -1

B. $(\frac{1}{\sqrt{2}})^{2011}$

C. 0

D. 1/2011

E. 1

14. Let a, b, and c be real numbers satisfying $a^2 + b^2 + c^2 = ab + bc + ac$. Among the statements

I. $a + b + c = 0$,

II. $c = a - b$,

III. $a = b = c$,

which is correct?

A. At least one of **I** and **II** must be true.

B. Only **III** must be true.

C. All three statements could be false.

D. Only **I** must be true.

E. All three statements must be true.

15. How many distinct real roots does the polynomial $x^3 - 2x^2 + 1 = 0$ have?

A. 0

B. 1

C. 2

D. 3

E. 4

16. A college student needs to walk from her dorm on one corner of an 80 ft $\times$ 150 ft rectangular lawn to her math class on the opposite corner of the lawn. She can walk at 5 ft/sec on the sidewalk around the outside of the lawn, or at 4 ft/sec diagonally across the grass. Which path is faster?

   A. She can save at least ten seconds by using the sidewalk.

   B. She can save less than ten seconds by using the sidewalk.

   C. Both paths are equally fast.

   D. She can save less than ten seconds by cutting across the grass.

   E. She can save at least ten seconds by cutting across the grass.

17. Let $A = (6, 0)$, $B = (0, 8)$, and $C = (0, 0)$. Find the equation of the circle passing through A, B, and C.

   A. $9x^2 + 9y^2 - 36x - 48y = 108$

   B. $x^2 + y^2 - 6x - 8y = 0$

   C. $x^2 + y^2 + 2xy - 6x - 8y = 0$

   D. $8x + 6y = 48$

   E. $16x^2 + 9y^2 = 576$

18. Let $a_1, a_2, \ldots, a_n, \ldots$ be a geometric sequence of real numbers. Suppose that $a_1 + a_2 = 1620$ and $a_1 + a_2 + \cdots + a_6 = 2340$. Find $a_1 + a_2 + \cdots + a_{10}$.

   A. 2346

   B. 2420

   C. 2440

   D. 2620

   E. 2916

19. Let ABC be a triangle such that both $\angle A$ and $\angle B$ are acute angles. Let $\overline{CD}$ be the altitude from C to $\overline{AB}$. If $\frac{AD}{DB} = \left(\frac{AC}{BC}\right)^2$, then, among the following statements,

   I. ABC is an isosceles triangle.

   II. ABC is a right triangle.

   III. ABC is an equilateral triangle.

   which is correct?

   A. Only **II** must be true.

   B. **I** and **II** must both be true.

   C. Only **I** must be true.

   D. **I** and **III** must both be true.

   E. At least one of **I** and **II** must be true.

20. Let S be the set of points in the plane which are exactly twice as far from $A = (2, 0)$ as they are from $B = (1, 1)$. Find the equation for S.

   A. $x^2 - y^2 - 3x + y + 2 = 0$

   B. $9x^2 + 9y^2 - 42x + 6y + 42 = 0$

   C. $3x^2 + 3y^2 - 4x - 8y + 4 = 0$

   D. $32x^2 + 32y^2 - 80xy - 56x + 88y + 11 = 0$

   E. $9x^2 + 9y^2 - 12x - 12y = 0$

21. The bacteria *polygeminus grex* has the property that a colony in a petri dish will double in size every three hours. If a single bacterium placed in a petri dish will reproduce to fill it in seventy-two hours, how long would it take two bacteria to fill the same dish?

   A. More than 60 hours.

   B. Between 50 and 60 hours.

   C. Between 40 and 50 hours.

   D. Between 30 and 40 hours.

   E. Less than 30 hours.

22. Find the area of the pentagon with vertices $(0,0)$, $(1,1)$, $(2,4)$, $(3,9)$, and $(4,16)$.

A. 10

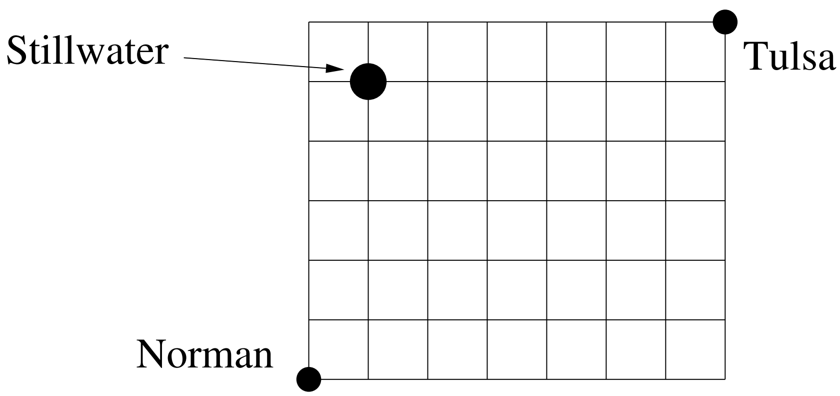
B. $10\frac{2}{3}$

C. 16

D. $21\frac{1}{3}$

E. 22

23. The Sooner Schooner wants to travel from Norman at $(0,0)$ to Tulsa at $(7,6)$. It can travel only directly North or East, and only in one-unit distances. Assuming it's not willing to pass through Stillwater at $(1,5)$, how many different paths can it take to Tulsa?



A. 42

B. 1674

C. 1703

D. 2011

E. 2053

24. Solve the inequality $x^{1+\log_{2011} x} > 2011x$.

A. $(2011, 2011^2)$

B. $(1/2011, 2011) \cup (2011^2, \infty)$

C. $(0, 1/2011) \cup (2011, \infty)$

D. $(2011, \infty)$

E. $(1/2011, 2011)$

25. Given a triangle ABC, let D be the point on $\overline{AB}$ such that $\frac{AD}{DB} = 3$, and let E be the point on $\overline{AC}$ such that $\frac{AE}{EC} = 4$. Let P be the intersection of lines CD and BE. If triangle BCP has area 1, find the area of triangle ABC.

A. 6

B. 8

C. 12

D. 16

E. 20

Part II. Team Round

1. Let ABC be a triangle with area 9. Prove that it is possible to subdivide ABC into nine smaller triangles each with area 1. Then prove that there is a point P on the interior of ABC such that every line through P divides ABC into two regions each with area between 4 and 5.

2. Factor the polynomial $p(x) = x^8 + x^4 + 1$ as a product of three nontrivial polynomials with integer coefficients. Describe the roots of p.

3. Suppose that a polygon P is invariant under the rotation about a given point c by an angle of $48°$. (This means the polygon obtained after the rotation coincides with P. For example, a square is invariant under a rotation about its center by $90°$, by not by $45°$.)

   (a.) Is P necessarily invariant under a rotation about c by $90°$?

   (b.) Is P necessarily invariant under a rotation about c by $72°$?

   (c.) Is P necessarily invariant under a rotation about c by $120°$?

4. The Fibonacci sequence is defined by $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all positive integers n. Prove that $F_{n+10} - F_n$ is divisible by 11 for all positive n.

5. Find all real numbers a such that the polynomial $x^{2011} - ax^{2010} + ax - 1$ is divisible by $(x-1)^2$.