Math 2163<br>Jeff Mermin's sections, Extra credit written project 5 due Monday, December 11 (at the final exam)

Preamble: This project is meant to unify several of the things we've learned at various parts of the course around the familiar idea of area.

It starts with some things that should be doable for high school students, and ends with complicated vector integrals. In order to fully understand what's going on, you'll have to do some independent reading from chapter 17.1 on Green's theorem.

As with the other written assignments, you may (and should) work in groups of up to five on this assignment. Every group member should sign the jointly written solution, signifying that you have all read the entire thing and approve of both the answers and the presentation.

If you choose to work in a group, my suggestion is that you plan to meet twice: once during dead week to get through the first four problems, then once over the weekend to finish.

In any event, all your solutions should be explained clearly, using complete mathematical sentences.

Consider the pentagon $P=A B C D E$, with vertices (in counterclockwise order) $A=(4,4), B=(10,0), C=(9,7), D=(2,10)$, and $E=(1,9)$.

1. Find the area of $P$ by chopping it into three triangles, and then using cross products to compute the areas of the triangles. (One issue that you'll need to explain: These points are all in $\mathbb{R}^{2}$, but cross products are only defined for vectors in $\mathbb{R}^{3}$. How do we get around that?)
2. If you had the same Algebra I textbook as me, you may have encountered an "enrichment" page that told you how to compute the area of any polygon: Make a matrix by stacking the vertices on top of each other in counterclockwise order (repeating the top vertex at the bottom), $M=\left[\begin{array}{cc}4 & 4 \\ 10 & 0 \\ 9 & 7 \\ 2 & 10 \\ 1 & 9 \\ 4 & 4\end{array}\right]$. Then set $D$ equal to the "sum of the downward diagonals", $D=(4)(0)+(10)(7)+(9)(10)+(2)(9)+(1)(4)$, and set $U$ equal to the "sum of the upward diagonals" similarly. The area of the polygon is $\frac{D-U}{2}$.
Interpret this recipe for our pentagon $P$, and verify that it gets the correct area.
3. Prove that the previous problem's technique works for an arbitrary triangle with vertices at $(a, b),(c, d)$, and $(e, f)$ by doing the algebra for both the cross product thing and the $\frac{D-U}{2}$ thing.
4. Use the fact that the $\frac{D-U}{2}$ technique works for triangles to prove that it works for quadrilaterals. (Hint: you don't need to do any cross product work here. Just chop a quadrilateral with vertices $(a, b),(c, d),(e, f)$, $(g, h)$ into triangles and look what happens when you compute the areas separately and add.)
5. Let $\mathcal{C}$ represent the perimeter of a polygon, traced once counterclockwise. Use Green's theorem to prove that the area is equal to $\int_{\mathcal{C}} \frac{x d y-y d x}{2}$. Then verify (by dividing the path into five line segments) that this integral produces the correct area for our pentagon $P$.
6. Explain the relationship between the $\frac{D-U}{2}$ technique and Green's theorem.
