6 A line and a plane

A line ℓ and a plane L are either parallel or intersecting. If they're parallel, there's a plane M containing ℓ which is parallel to L. We'd also like to know the distance from ℓ to L. If they intersect, we'd like to know the point and angle of intersection.

6.1 The intersecting case

Let ℓ be defined by the parametrization $(x, y, z) = (-1, 2, 2,) + t(5, 4, 5)$ and let L be defined by the equation $3x + 5y - z = 3$.

1. Verify that ℓ is not contained in L.

Notice that ℓ contains the point $(-1,2,2)$ (when $t=0$),

but L does not: $-3(-1)+5(2)-2 = 11 \neq 3$.

2. Find the point of intersection.

The point of intersection $P=(x,y,z)$ is on ℓ, so there's some t such that $\left\{\begin{array}{l} x=-1+5t \\ y=2+4t \\ z=2+5t \end{array}\right\}$.

But P is also on L, so $3x+5y-z=3$.

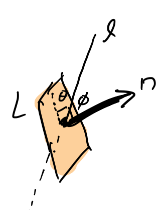
Substituting, $3(-1+5t)+5(2+4t)-(2+5t)=3$.

We conclude that $t=\frac{-1}{15}$.

Thus $P=\left(-1+5\left(\frac{-1}{15}\right), 2+4\left(\frac{-1}{15}\right), 2+5\left(\frac{-1}{15}\right)\right)$
$= \boxed{\left(-\frac{4}{3}, \frac{26}{15}, \frac{5}{3}\right)}$

3. Find the angle of intersection.

Let θ be the desired angle. Then θ is complementary to ϕ, the angle between ℓ and a normal vector to L.



We have $n=\langle 3,5,-1\rangle$, and ℓ has direction vector $v=\langle 5,4,5\rangle$.

Thus $\cos\phi = \frac{n\cdot v}{\sqrt{(n\cdot n)(v\cdot v)}} = \frac{30}{\sqrt{2310}} \left(=\sqrt{\frac{30}{77}}\right)$.

Since θ and ϕ are complements, we get

$\sin\theta = \cos\phi = \sqrt{\frac{30}{77}}$, i.e. $\boxed{\theta = \arcsin\left(\sqrt{\frac{30}{77}}\right)}$.

6.2 The parallel case

Let ℓ be defined by the parametrization $(x, y, z) = (-2, -3, 2) + t(-1, 5, 2)$ and let L be defined by the equation $x + 3y - 7z = -2$.

1. Verify that ℓ and L are parallel.

Let $v = \langle -1, 5, 2 \rangle$ (a direction vector for ℓ)
and $n = \langle 1, 3, -7 \rangle$ (a normal vector for L).

Notice that $v \bullet n = -1 + 15 - 14 = 0$, so v and n are perpendicular.

Thus either ℓ is inside L or ℓ is parallel to L.
But $(-2,-3,2) \in \ell$ and $(-2,-3,2) \notin L$ (since $-2+3(-3)-7(2) = -25 \neq -2$)
so ℓ isn't inside L.

We conclude that ℓ and L are parallel.

2. Find an equation for M.

Since $-2+3(-3)-7(2) = -25$, (and $-t + 3(5t) - 7(2t) = 0$ for all t),

we have that all points on ℓ satisfy

$$\boxed{x + 3y - 7z = -25.}$$

The plane $M: x+3y-7z = -25$ is parallel to L since they have the same normal vector.

3. Find the distance from ℓ to L.

This is the same as the distance from L to M,

which is $\dfrac{|(-25) - (-2)|}{\| \langle 1, 3, -7 \rangle \|} = \boxed{\dfrac{23}{\sqrt{59}}}$.