Math 2163

Jeff Mermin's section, Test 1, September 20

On the essay questions (# 4–10) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

Do not evaluate any integrals on this test. If you would take an integral, instead simplify the integrand and the limits of integration (if any), and leave the integral as your final answer.

1. (**30 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   In the statements below, a is a real number, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^3$, t is a parameter, x, y, and z are variables which are each differentiable functions of t, $\mathbf{r} = \mathbf{r}(t) = (x(t), y(t), z(t))$ is a parametrization of a curve, and P is a point in $\mathbb{R}^3$.

   (a) $|\mathbf{v} \bullet \mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|$.

   True. ($v \cdot w = |v||w| \cos\theta$)

   (b) $\mathbf{v} - \mathbf{w} = \mathbf{w} - \mathbf{v}$.

   False.

   (c) $\dfrac{d}{dt}\|\mathbf{r}\| = \left\|\dfrac{d\mathbf{r}}{dt}\right\|$

   False.

   (d) $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$.

   True.

   (e) $(\mathbf{u} \bullet \mathbf{v})\mathbf{w} = \mathbf{u}(\mathbf{v} \bullet \mathbf{w})$.

   False. (Try $u = \langle 1,0,0\rangle$, $v = \langle 1,0,0\rangle$, $w = \langle 0,0,1\rangle$.)

   (f) The equations $(x, y, z) = P + t\mathbf{v}$ and $(x, y, z) = P - t\mathbf{v}$ define the same line.

   True.

   (g) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$.

   False. (Try $u = \langle 1,0,0\rangle$, $v = \langle 1,0,0\rangle$, $w = \langle 0,1,0\rangle$.

   (h) The equations $x = 2$, $y = -1$, $z = 0$ define a line in $\mathbb{R}^3$.

   False. ($(2,-1,0)$ is a point.)

   (i) $\mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{v}$.

   False.

   (j) $\displaystyle\int_{t=0}^{t=1} \frac{d\mathbf{r}}{dt} dt = \mathbf{r}(1) - \mathbf{r}(0)$.

   True.

2. (**20 points**) Let $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in $\mathbb{R}^3$. Are the following expressions vectors, scalars, or nonsense? (No justification is necessary on this problem, but wrong answers with good explanations may receive credit.)

(a) $(((\mathbf{v} \times \mathbf{w}) - \mathbf{x}) - \mathbf{y}) \times \mathbf{z}$

This is a vector.

(b) $(\mathbf{v} \bullet ((\mathbf{w} \bullet \mathbf{x})\mathbf{y}))\mathbf{z}$

This is a vector.

(c) $(\mathbf{v} \bullet (\mathbf{w} - (\mathbf{x} + \mathbf{y}))) \bullet \mathbf{z}$

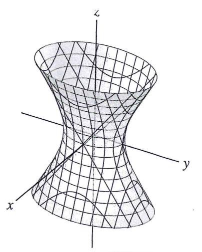
This is nonsense. (You can't dot a scalar with a vector.)

(d) $(\mathbf{v} \bullet \mathbf{w})(\mathbf{x} \bullet (\mathbf{y} \times \mathbf{z}))$

This is a scalar.

3. (**20 points**) For each of the quadric surfaces below, identify it by type (ellipsoid, hyperbolic cylinder, etc.) **or** suggest a possible equation. (No justification is necessary on this problem, but wrong answers with good explanations may receive credit.)

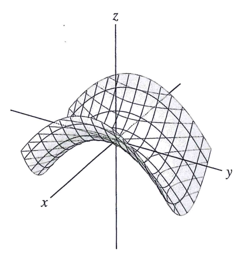
(a)



This is a hyperboloid of one sheet.

The equation might be $x^2+y^2-z^2=1$.

(b)

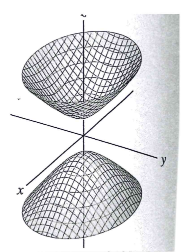


This is a hyperbolic paraboloid.

The equation might be

$z=x^2-y^2$.

(c)

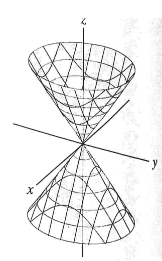


This is a hyperboloid of two sheets.

The equation might be

$x^2+y^2-z^2=-1$

(d)



This is a cone.

The equation might be $x^2+y^2-z^2=0$.

4. (**20 points**) Let $\mathbf{x} = \langle -2, -2, 4\rangle$ and $\mathbf{y} = \langle 0, 3, -1\rangle$. Compute the following:

   (a) $\mathbf{x} - 3\mathbf{y}$. $= \langle -2, -2, 4\rangle - \langle 0, 9, -3\rangle$

   $= \langle -2-0, -2-9, 4--3\rangle$

   $= \langle -2, -11, 7\rangle$.

   (b) $\mathbf{x} \bullet \mathbf{y}$. $= (-2)(0) + (-2)(3) + (4)(-1)$

   $= -10$

   (c) $\mathbf{x} \times \mathbf{y}$. $= \langle (-2)(-1) - (4)(3), (4)(0) - (-2)(-1), (-2)(3) - (-2)(0)\rangle$

   $= \langle -10, -2, -6\rangle$

   (d) $(\mathbf{x} - 3\mathbf{y}) \times \mathbf{y}$. $= x \times y - 3(y \times y)$

   $= x \times y - 0$

   $= \langle -10, -2, -6\rangle$.

5. (**10 points**) Find a parametrization of the line passing through the points $P = (0, 1, -1)$ and $Q = (-1, 5, 2)$.

The direction vector is $\vec{PQ} = \langle -1-0, 5-1, 2-{}^{-}1 \rangle$

$= \langle -1, 4, 3 \rangle$.

So a parametrization is

$(x, y, z) = (0, 1, -1) + \langle -1, 4, 3 \rangle t$.

6. (**15 points**) Find an equation for the plane containing the point $R = (-3, 2, 3)$ and perpendicular to the line $(x, y, z) = (2, -1, 0) + (2, 3, 3)t$.

The line's direction vector $\langle 2, 3, 3 \rangle$ is normal to the plane.

Thus the plane has equation

$\boxed{2(x - {}^{-}3) + 3(y-2) + 3(z-3) = 0}$

or $\boxed{2x + 3y + 3z = 9}$

7. (**15 points**) Find a parametrization of the line of intersection of the planes $G : x - 3y + 4z = 1$ and $H : 3x + 2y - 2z = -6$.

To find a point in the intersection, we invent a third equation and solve:

$$\left\{\begin{array}{l} x - 3y + 4z = 1 \\ 3x + 2y - 2z = -6 \\ x = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} -3y + 4z = 1 \\ 2y - 2z = -6 \end{array}\right\} \Rightarrow \begin{array}{l} y = -11 \\ z = -8 \end{array}$$

So $(0, -11, -8)$ is in our line.

Alternatively, find a second point by inventing a different third equation in place of $x = 0$.

To find a direction, observe that $n_F \times n_G = \langle -2, 14, -7 \rangle$ is perpendicular to both normal vectors, so parallel to both planes. Thus it's parallel to the line of intersection, and it's parametrized by $(x, y, z) = (0, -11, -8) + \langle -2, 14, -7 \rangle t$.

8. (**20 points**) Let C be the curve defined by the parametrization $\mathbf{r}(t) = (e^t, t \ln t, t^2 - t)$. Find a parametrization of the tangent line to C at the point $P = (e, 0, 0)$.

At P, t satisfies $\left\{\begin{array}{l} e^t = e \\ t \ln t = 0 \\ t^2 - t = 0 \end{array}\right\}$ so $t = 1$.

The tangent vector is $\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$

$$= \langle e^t, \ln t + 1, 2t - 1 \rangle;$$

at $t = 1$ this is $\langle e, 1, 1 \rangle$.

Thus the tangent line is parametrized by

$$(x, y, z) = (e, 0, 0) + \langle e, 1, 1 \rangle (t - 1)$$

(Use $t - 1$ instead of t so that it passes through P at $t = 1$, just like the curve.)

9. (**Extra credit: 10 points**) Let S be the surface defined by the equation (in spherical coordinates) $\rho = \cos\theta \sin\phi$. Find (and simplify, if necessary) an equation for S in the form $F(x, y, z) = 0$, and describe S verbally.

Rewriting, we get $\rho = \frac{x}{r}\frac{r}{\rho}$

$$\rho^2 = x$$

$$x^2 + y^2 + z^2 = x$$

$$\boxed{(x^2 - x) + y^2 + z^2 = 0}$$

$$x^2 - x + \frac{1}{4} + y^2 + z^2 = \frac{1}{4}$$

$$(x - \tfrac{1}{2})^2 + y^2 + z^2 = \frac{1}{4}$$

This is the sphere with radius $\frac{1}{2}$, centered at $(\frac{1}{2}, 0, 0)$.

10. (**Extra credit: 10 points**) Let F be the plane defined by the equation $2x - y + 2z = 4$. Find another plane G which is parallel to F but exactly two units away.

We want $2x - y + 2z = D$,

with $\dfrac{|D-4|}{|\langle 2,-1,2\rangle|} = 2$, i.e.

$$|D-4| = 6$$

Thus G is either $2x - y + 2z = 10$

or $2x - y + 2z = -2$.