Math 2163

Jeff Mermin's sections, Final exam,December 16

On the essay questions (# 2–11) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

I hope you won't need this integral table, but here it is anyway.

1. (**40 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   In the statements below, **a**, **b**, **c**, **x**, and **y** are vectors, a, b, c, d, d', and L are numbers, $x(t)$, $y(t)$, and $z(t)$ are twice continuously differentiable functions on **R**, $C : \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a curve with associated frame **T**, **N**, **B**, f and g are continuous functions with continuous partial derivatives of all orders on their domains, which include all of $\mathbb{R}^2$ or $\mathbb{R}^3$ except possibly the origin, D and R are simply connected regions inside the interiors of these domains, and F is a vector field on $\mathbb{R}^2$ or $\mathbb{R}^3$.

   $x, y, z, r, \theta, \rho, \phi$ are the usual rectangual, cylindrical, and spherical coordinates.

   (a) If $u = 2x$ and $v = 2y$, then $\iint\limits_R f dxdy = 2 \iint\limits_R f dudv$.

   False. $du\, dv = 4\, dx dy$.

   (b) If R is the sphere of radius one about the origin, then $\iiint\limits_R f dV =$

   $$\int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=1} f \rho^2 \sin\phi \, d\rho d\theta d\phi.$$

   False. It should be $\int_{\phi=0}^{\phi=\pi}$

   (c) $\iint\limits_R f(x,y) + g(x,y) dA = \iint\limits_R f(x,y) dA + \iint\limits_R g(x,y) dA.$

   True.

   (d) If f has a local maximum at $(0,0)$, then $f_x(0,0) = 0$.

   True.

   (e) $\frac{d\mathbf{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$.

   True.

   (f) The angle between two planes is equal to the angle between their normal vectors.

   True.

   (g) $\iint\limits_R f\, dxdy = \iint\limits_R f\, drd\theta$.

   False. $dx dy = r dr d\theta$

(h) $\iiint_R f(x,y,z)dxdydz = \iiint_R f(x,y,z)dzdxdy$.

True.

(i) $\mathbf{x} - \mathbf{y} = \mathbf{y} - \mathbf{x}$.

False.

(j) If R_1 and R_2 are disjoint regions and $R = R_1 \cup R_2$ is their union, then $\iiint_R f(x,y,z)dV = \iiint_{R_1} f(x,y,z)dV + \iiint_{R_2} f(x,y,z)dV$.

True.

(k) $\mathbf{N} = \mathbf{B} \times \mathbf{T}$.

True.

(l) If $R = \{(x,y,z) : x^2 + y^2 + z^2 \leq 9\}$, then $\iiint_R dV = 36\pi$.

True. R is a sphere of radius 3; its volume is $\frac{4}{3}\pi(3)^3$.

(m) Suppose $f(a,b,c) = 0$. Then $\nabla f(a,b,c)$ is normal to the surface $f(x,y,z) = 0$ at the point (a,b,c).

True. If $\langle dx, dy, dz \rangle$ is a tangent vector, then $0 = df = f_x dx + f_y dy + f_z dz$
$= \langle dx, dy, dz \rangle \cdot \nabla f$.

(n) $a(b\mathbf{x}) = (ab)\mathbf{x}$.

True.

(o) If f has an absolute maximum on the region D at (a,b), then f has a local maximum at (a,b).

False. (a,b) could be on the boundary of D.

(p) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

False. (Try $a = b = \langle 1,0,0 \rangle$, $c = \langle 0,1,0 \rangle$).

(q) If $\lim_{(x,y)\to(a,b)} f(x,y) = L$, then $\lim_{(x,y)\to(a,b)} (\cos f(x,y)) = \cos L$.

True.

(r) If z is defined implicitly as a function of x and y by $f(x,y,z) = 0$, then

$$\frac{\partial z}{\partial x} = \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}.$$

False. $\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}$.

(s) If $\int_C F \bullet d\mathbf{r} = 0$ for all closed curves C, then F is conservative.

True. (A potential function is $P(x,y,z) = \int_D F \bullet dr$ where D is <u>any</u> curve from $(0,0,0)$ to (x,y,z).)

(t) The distance between the planes $F : ax + by + cz = d$ and $G : ax + by + cz = d'$ is $|d - d'|$.

False. It's $\dfrac{|d-d'|}{|\langle a,b,c\rangle|}$

2. (**20 points**)

Let $\mathbf{x} = \langle 2, -4, 3\rangle$ and $y = \langle -1, 4, 0\rangle$. Compute the following.

(a) $\mathbf{x} + \mathbf{y}$. $= \langle 2+-1, -4+4, 3+0\rangle$

$= \langle 1, 0, 3\rangle$

(b) $\mathbf{x} \bullet \mathbf{y}$. $= (2)(-1) + (-4)(4) + (3)(0)$

$= -18$.

(c) $\mathbf{x} \times \mathbf{y}$. $= \langle (-4)(0) - (3)(4), (3)(-1) - (2)(0), (2)(4) - (-4)(-1)\rangle$

$= \langle -12, -3, 4\rangle$.

(d) $\text{proj}_{\mathbf{y}}(\mathbf{x})$, the vector projection of $\mathbf{x}$ onto $\mathbf{y}$.

$\text{proj}_y(x) = \left(\dfrac{x \cdot y}{y \cdot y}\right) y = \dfrac{-18}{17} y = -\dfrac{18}{17}\langle -1, 4, 0\rangle = \langle \dfrac{18}{17}, \dfrac{72}{17}, 0\rangle$.

3. (**10 points**) Find an equation for the plane containing the parallel lines $(x, y, z) = (-5, -1, -1) + (1, 1, 4)s$ and $(x, y, z) = (3, -4, 3) + (1, 1, 4)t$.

Two vectors in the plane are $\langle 1,1,4\rangle$ and $(3,4,3) - (-5,-1,-1)$
$= \langle 8,5,4\rangle$.

Thus $\langle 1,1,4\rangle \times \langle 8,5,4\rangle = \langle -16, 28, -3\rangle$ is normal.

Since the plane goes through $(3,-4,3)$, its equation is

$$-16x + 28y - 3z = -16(3) + 28(4) - 3(3)$$

$$\boxed{-16x + 28y - 3z = 55}$$

4. (**10 points**) Compute $\displaystyle\int_{x=0}^{x=2}\int_{y=0}^{y=\sqrt{4-x^2}}\int_{z=0}^{z=2} x^2 z + y^2 z\, dz dy dx.$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=\sqrt{4-x^2}} \left[\frac{1}{2}x^2z^2 + \frac{1}{2}y^2z^2\right]_{z=0}^{z=2} dy\, d\cancel{x}$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=\sqrt{4-x^2}} 2x^2 + 2y^2 \, dy\, dx$$

$$= \int_{r=0}^{r=2}\int_{\theta=0}^{\theta=\frac{\pi}{2}} 2r^2 \, r\, dr\, d\theta$$

$$= \int_{r=0}^{r=2} \left[2r^3\theta\right]_{\theta=0}^{\theta=\frac{\pi}{2}} dr$$

$$= \int_{r=0}^{r=2} \pi r^3 \, dr$$

$$= \left[\frac{\pi}{4}r^4\right]_{r=0}^{r=2} = \boxed{4\pi}.$$

5. (**10 points**) Find the point on the plane $x - y + z = 5$ that is closest to the point $(1, 2, 3)$.

This is where the normal line $(x,y,z) = (1,2,3) + \langle 1,-1,1\rangle t$ meets the plane.

So $(1+t) - (2-t) + (3+t) = 5$
$2 + 3t = 5$
$t = 1.$

We get $(x,y,z) = (1,2,3) + \langle 1,-1,1\rangle$
$= \boxed{(2,1,4)}$.

6. (**10 points**) Find all critical points of the function $f(x, y) = x^3 - y^3 + 3xy - 6$.

We need $d_x = d_y = 0$.

$d_x = 3x^2 - 3y$, $d_y = -3y^2 + 3x$.

So $3x^2 - 3y = 0 \Rightarrow y = x^2$, so $-3y^2 + 3x = 0$
$\Rightarrow -3x^4 + 3x = 0$
$-3(x^4 - x) = 0$
$-3(x)(x^3 - 1) = 0$

So $x = 0$ or $x^3 - 1 = 0$
$\Downarrow$ $\Downarrow$
$y = 0$ $x = 1, y = 1$

The critical points are $(0,0)$ and $(1,1)$.

7. (**20 points**) Let $f(x, y, z) = xyz$.

(a) Find the tangent plane to the level surface of f at the point $(1, 2, 3)$.

The level surface is $f = 6$.

We get $df = d6 = 0$.

By the chain rule, $f_x dx + f_y dy + f_z dz = 0$.

Near $(1,2,3)$, we have $f_x = yz = 6$, $f_y = xz = 3$, $f_z = xy = 2$

$dx = (x-1)$, $dy = (x-2)$, $f_z = (x-3)$.

Thus the tangent plane is $6(x-1) + 3(x-2) + 2(x-3) = 0$.

(b) Find the directional derivative $D_{\mathbf{u}}(f)(1, 2, 3)$ if $\mathbf{u} = \langle 2, 1, 2 \rangle$.

$D_u(f) = 2f_x + 1f_y + 2f_z$

$= 2yz + xz + 2xy$

$D_u(f)(1,2,3) = 12 + 3 + 4 = \boxed{19}$.

(Alternatively, $\hat{u} = \langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \rangle$, so $D_{\hat{u}}(f)(1,2,3) = \frac{12}{3} + \frac{3}{3} + \frac{4}{3}$

$= \boxed{\frac{19}{3}}$.

8. (**20 points**) Let $F(x, y, z) = \langle y^2 \cos z, 2xy \cos z, -xy^2 \sin z\rangle$.

(a) Is F conservative? If so, find a potential function $f(x, y, z)$ satisfying $F = \nabla f$. If not, explain why not.

If $F = \langle f_x, f_y, f_z \rangle$, then $f_{xy} = 2y \cos z$, $f_{yx} = 2y \cos z$
$f_{xz} = -y^2 \sin z$, $f_{zx} = -y^2 \sin z$
$f_{yz} = -2xy \sin z$, $f_{zy} = -2xy \sin z$.

Since these agree, F is conservative.

We have $f = \int y^2 \cos z \, dx = xy^2 \cos z + C_x$,
$f = \int 2xy \cos z \, dy = xy^2 \cos z + C_y$,
and $f = \int -xy^2 \sin z \, dz = xy^2 \cos z + C_z$.

Merging these, $f = xy^2 \cos z + C$, so $\boxed{f = xy^2 \cos z}$ will do.

(b) Compute $\displaystyle\int_C F \bullet d\mathbf{r}$, where C moves from the origin to $(0, 0, 2\pi)$ along the curve $(x, y, z) = (\sin z, 1 - \cos z, z)$.

Since F is conservative, the integral is

$f(0,0,2\pi) - f(0,0,0)$
$= 0 - 0 = \boxed{0}$.

9. (**10 points**) Write down an iterated integral which expresses the mass of the parabolic dome $z \leq 1 - x^2 - y^2$, $z \geq 0$, if it has density $\rho(x, y, z) = 3 - z$. **Do not evaluate**.

This is $0 \leq z \leq 1 - r^2$, which requires $0 \leq r \leq 1$ (and $0 \leq \theta \leq 2\pi$).

So our integral is

$$\int_{r=0}^{r=1} \int_{z=0}^{z=1-r^2} \int_{\theta=0}^{\theta=2\pi} (3-z)(r)\, d\theta\, dz\, dr.$$

10. (**Extra Credit: 10 points**) Write down an iterated integral expressing the surface area of the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$. **Do not evaluate**.

This is $\iint_R$ (area above the square with corners (x,y), $(x+dx,y)$, $(x,y+dy)$, $(x+dx,y+dy)$)

$$= \iint_R \left(\left| \langle dx, 0, z_x dx \rangle \times \langle 0, dy, z_y dy \rangle \right| \right)$$

$$= \iint_R \left| \langle -z_x dx\, dy, -z_y dx\, dy, dx\, dy \rangle \right|$$

$$= \iint_R \sqrt{z_x^2 + z_y^2 + 1}\, dx\, dy = \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy$$

$$= \boxed{\int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} \sqrt{1+4r^2}\, (r\, d\theta\, dr)}$$

11. (**Extra Credit: 10 points**) In the space remaining on this page, write the largest integer you can. You do not have to simplify, as, for example, "10^3" may take up less space than "1000". However, I should be able (at least in theory) to determine the precise value of your number with no ambiguity or reference to the experiential universe (so, for example, "the number of stars in the sky" is right out), so you may need to use some space defining your notation.