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Math 2163

Jeff Mermin's sections, Test 3, December 2

On the essay questions (# 2–6) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

1. (**30 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   In the statements below, **a**, **b**, and **c** are vectors in $\mathbb{R}^3$, a, b, c, and d are real numbers, $C : \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a curve in $\mathbb{R}^3$ with associated vectors **T**, **N**, and **B**, R is a closed and bounded region in $\mathbf{R}^2$ or $\mathbf{R}^3$, and f is a function on R with continuous partial derivatives of all orders.

   (a) The area of R is $\displaystyle\iint_R 1dA$.

   True.

   (b) $(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$.

   True.

   (c) $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{a}$.

   False.

   (d) $\mathbf{N} = \mathbf{B}\times\mathbf{T}$.

   True.

   (e) If (a,b) is a critical point of f, $f_{xx}(a,b) = 1$, and $f_{yy}(a,b) = -1$, then (a,b) is a saddle point of f.

   True. $D = (1)(-1) - [f_{xy}(a,b)]^2 \leq -1 < 0$.

   (f) $\mathbf{a}-\mathbf{b} = \mathbf{b}-\mathbf{a}$.

   False.

   (g) If $f(x,y) \geq 0$ for all $(x,y)\in R$, then $\displaystyle\iint_R f dA \geq 0$.

   True.

   (h) The distance between the planes $F : ax+by+cz = d$ and $G : ax+by+cz = d'$ is $|d-d'|$.

   False. It's $\dfrac{|d-d'|}{\sqrt{a^2+b^2+c^2}}$

   (i) If $R = \{(x,y,z) : 1 \leq x^2+y^2+z^2 \leq 16\}$, then $\displaystyle\iiint_R dV = 84\pi$.

   True. R is a sphere of radius 4 with a sphere of radius 1 removed. Its volume is $\frac{4}{3}\pi(4)^3 - \frac{4}{3}\pi(1)^3 = 84\pi$.

   (j) $\frac{d\mathbf{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle$.

   True.

2. (**30 points**) Evaluate the integrals:

(a) $\displaystyle\int_{x=1}^{x=4}\int_{y=0}^{y=2} 4x^3 + 3y^2\,dydx$

$= \displaystyle\int_{x=1}^{x=4} \left[4x^3y - y^3\right]_{y=0}^{y=2} dx$

$= \displaystyle\int_{x=1}^{x=4} (8x^3 - 8) - 0 \quad dx$

$= \left[2x^4 - 8x\right]_{x=1}^{x=4} = (512 - 32) - (2 - 8)$

$= \boxed{486}.$

(b) $\displaystyle\int_{x=0}^{x=4}\int_{y=0}^{y=x} \sqrt{xy}\,dydx = \int_{x=0}^{x=4}\int_{y=0}^{y=x} x^{1/2}y^{1/2}\,dy\,dx$

$= \displaystyle\int_{x=0}^{x=4} \left[\frac{2}{3}x^{1/2}y^{3/2}\right]_{y=0}^{y=x} d\cancel{x}$

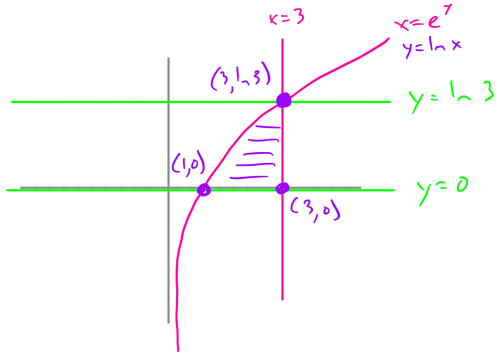
$= \displaystyle\int_{x=0}^{x=4} \frac{2}{3}x^2 - 0 \quad dx$

$= \left[\frac{2}{9}x^3\right]_{x=0}^{x=4} = \frac{128}{9} - 0$

$= \boxed{\frac{128}{9}}$

3. (**20 points**) Consider the iterated integral $I = \int_{y=0}^{y=\ln 3} \int_{x=e^y}^{x=3} \frac{1}{\ln x} dx dy$.

(a) Reverse the order of integration.



We get

$$\int_{x=1}^{x=3} \int_{y=0}^{y=\ln x} \frac{1}{\ln x} \, dy \, dx$$

(b) Evaluate I, using whichever order seems easier.

$\int \frac{1}{\ln x} dx$ seems hard, so we try the new one.

$$\int_{x=1}^{x=3} \int_{y=0}^{y=\ln x} \frac{1}{\ln x} \, dy \, dx = \int_{x=1}^{x=3} \left[\frac{y}{\ln x} \right]_{y=0}^{y=\ln x} dx$$

$$= \int_{x=1}^{x=3} \left(\frac{\ln x}{\ln x} - \frac{0}{\ln x} \right) dx$$

$$= \int_{x=1}^{x=3} 1 \, dx$$

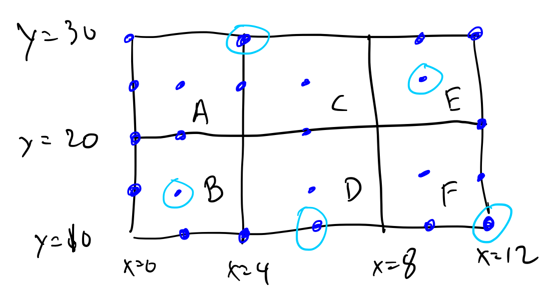
$$= \left[x \right]_{x=1}^{x=3} = 3 - 1 = \boxed{2}.$$

4. (**10 points**) Some values of a continuous function $f(x, y)$ on the rectangle $R = \{0 \leq x \leq 12, 10 \leq y \leq 30\}$ are given in the table below. (Apparently f is hard to compute, because some values are unknown). Estimate the value of $\iint_R f(x, y)dA$ using a Riemann sum with at least six summands.

Warning: The $x = 8$ column is missing.

$y \backslash x$	0	2	4	6	10	12
10	?	−10	−10 D	−9	−6	0 F
15	−7	−4 B	?	2	4	8
20	−1	3	?	8	?	11
25	−3	−1	−1	−1	0 E	?
30	7	?	10 A, C	?	11	13

Chop R into 6 pieces:



The dots represent known information about f, from the table.

We choose one from each region.

Region	A	B	C	D	E	F
dx	4	4	4	4	4	4
dy	10	10	10	10	10	10
Sample point	(4,30)	(2,15)	(4,30)	(6,10)	(10,25)	(12,10)
Sample value of f	−10	−4	−10	−10	0	0
dA = dx dy	40	40	40	40	40	40
Summand	(−10)(40)	(−4)(40)	(−10)(40)	(−10)(40)	(0)(40)	(0)(40)

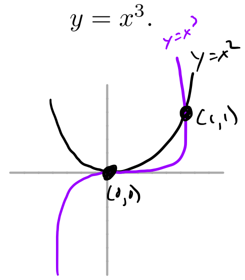
We approximate the integral as $(-10)(40) + (-4)(40) + (-10)(40) + (-10)(40) + (0)(40) + (0)(40)$

$= (-34)(40) = -136.$

(Of course, you have lots of choices for how to chop R into (at least) six pieces, and lots of choices for sample points within each region.)

5. (**60 points**) Express the following as iterated integrals, using a coordinate system of your choice.

(a) $\displaystyle\iint_R x + y\,dA$, where R is the region between the curves $y = x^2$ and $y = x^3$.



$$\int_{x=0}^{x=1}\int_{y=x^3}^{y=x^2} x+y \; dy\; dx \qquad \text{works.}$$

(b) The mass of the hemisphere $R = \{(x, y, z) : x^2 + y^2 + z^2 \le 9, z \ge 0\}$, if its density at the point (x, y, z) is $\underbrace{25 - x^2 - y^2}$.

R is a hemisphere, which seems to call for spherical coordinates.

This is $25-r^2$, which seems to call for cylindrical coordinates.

Using spherical coordinates, $0 \le \rho \le 3$; $0 \le \theta \le 2\pi$, $0 \le \phi \le \frac{\pi}{2}$ and the density is $25-(\rho\sin\phi)^2$.

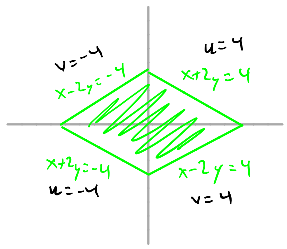
So we get $$\int_{\rho=0}^{\rho=3}\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\frac{\pi}{2}} (25-\rho^2\sin^2\phi)\left(\rho^2\sin\phi \; d\phi\, d\theta\, d\rho\right).$$

Using cylindrical coordinates, $0 \le r \le 3$, $0 \le z \le \sqrt{9-r^2}$, $0 \le \theta \le 2\pi$ (or $0 \le z \le 3$, $0 \le r \le \sqrt{9-z^2}$) and the density is $25-r^2$.

So we get $$\int_{r=0}^{r=3}\int_{\theta=0}^{\theta=2\pi}\int_{z=0}^{z=\sqrt{9-r^2}} (25-r^2)\left(r\, dz\, d\theta\, dr\right). \qquad \text{(or similar).}$$

Using rectangular coordinates, we get one of six options, which will be similar to $$\int_{x=-3}^{x=3}\int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}}\int_{z=0}^{z=\sqrt{9-x^2-y^2}} (25-x^2-y^2)(dz\,dy\,dx).$$

6. $\displaystyle\iint_R \frac{x+2y}{x-2y}\,dA$, where R is the region $\{(x,y) : |x| + |2y| \le 4\}$.



Set $u = x+2y$, $v = x-2y$.

Our integral is thus

$$\int_{u=-4}^{u=4}\int_{v=-4}^{v=4} \frac{u}{v}\,dA,$$

and $dA = dx\,dy = \frac{dx\,dy}{du\,dv}\,du\,dv$.

$\frac{dx\,dy}{du\,dv}$ is the Jacobian $\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$, which we can't compute without solving for x, y in terms of u, v.

But $\frac{dx\,dy}{du\,dv} = \frac{1}{\frac{du\,dv}{dx\,dy}}$, and $\frac{du\,dv}{dx\,dy} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$= \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = |-2-2|$
$= 4$.

So $\frac{dx\,dy}{du\,dv} = \frac{1}{4}$, and we get

$$\int_{u=-4}^{u=4}\int_{v=-4}^{v=4} \left(\frac{u}{v}\right)\left(\frac{1}{4}\right) dv\,du.$$

7. (**Extra credit: 20 points**) The *standard normal density* or *bell curve* is very important in probability and statistics, where it is used to describe certain kinds of repeatable experiments. The bell curve is given by the function $f(x) = \frac{1}{A}e^{-\frac{x^2}{2}}$, where A is chosen to make the bell curve into a probability density, that is, $\int_{x=-\infty}^{x=\infty} f(x)\,dx = 1$.

Thus $A = \int_{x=-\infty}^{x=\infty} e^{-\frac{x^2}{2}}\,dx$. Unfortunately, it is known that the indefinite integral $\int e^{-\frac{x^2}{2}}\,dx$ cannot be evaluated algebraically. However, we can still solve for the exact value of A.

Let $B = \int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty} e^{-\frac{x^2}{2}}e^{-\frac{y^2}{2}}\,dydx$.

(a) Describe the relationship between A and B.

$$B = \int_{x=-\infty}^{x=\infty} e^{-\frac{x^2}{2}}\,dx \int_{y=-\infty}^{y=\infty} e^{-\frac{y^2}{2}}\,dy = A^2.$$

(b) Rewrite B using polar coordinates.

$$B = \int_{r=0}^{r=\infty}\int_{\theta=0}^{\theta=2\pi} e^{-\frac{r^2}{2}}\,r\,d\theta\,dr$$

(c) Find A.

$A = \sqrt{B}$, and $B = \int_{r=0}^{r=\infty} 2\pi r\, e^{-\frac{r^2}{2}}\,dr$

$$= \left[-2\pi e^{-\frac{r^2}{2}}\right]_{r=0}^{r=\infty} = (0) - (-2\pi) = 2\pi$$

So $A = \sqrt{2\pi}$.