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Math 2163

Jeff Mermin's sections, Test 2, November 2

On the essay questions (# 2–8) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

1. (**30 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   In the statements below, $C : \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a curve, (a, b) is a point in the closed and bounded region D, $f(x, y)$ is a continuous function defined on some region including D, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors, $\mathbf{u}$ is a unit vector, and L is a number.

   (a) $\frac{d\mathbf{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$.

   True.

   (b) If $\lim_{x\to 0} f(x,0) = \lim_{y \to 0} f(0,y) = L$, then $\lim_{(x,y)\to(0,0)} f(x,y) = L$.

   False. (Consider $f = \frac{xy}{x^2+y^2}$).

   (c) $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{b})(\mathbf{a} \bullet \mathbf{c})$.

   False.

   (d) The angle between two planes is equal to the angle between their normal vectors.

   True.

   (e) $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.

   True.

   (f) If $\lim_{(x,y)\to(a,b)} f(x,y) = L$, then $\lim_{(x,y)\to(a,b)} \cos f(x,y) = \cos L$.

   True.

   (g) If f has an absolute maximum on the region D at (a, b), then f has a local maximum at (a, b).

   False. (This is true if (a,b) is on the interior of D, but not on the boundary.)

   (h) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

   False. (Try $a = b = \langle 1,0,0\rangle$, $c = \langle 0,1,0 \rangle$)

   (i) Two lines define a plane.

   False. (They could be skew.)

   (j) If $f(x, y) = x + y$, then $|D_{\mathbf{u}}(f)(x, y)| \le 2$ for all x and y.

   True. $f_x = 1$, $f_y = 1$, so if $u = (a,b)$ then $D_u(f) = a(1) + b(1)$ $= a + b$.

   and $a, b \le 1$ since u is a unit vector.

2. (**15 points**) Prove that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+y}$$

does not exist.

Approaching along $y=0$, we have $\lim_{x\to 0} \frac{x^2}{x^2} = 1$.

Along $x=0$, we have $\lim_{y\to 0} \frac{y^2}{y} = 0$.

These disagree.

3. (**15 points**) Find an equation for the tangent plane to the graph of the function $z = \frac{x+y}{4x}$ at the point $(1,3,1)$.

$z_x = \frac{4x - 4(x+y)}{16x^2} = \frac{-y}{4x^2}$ and $z_y = \frac{1}{4x}$.

At $(1,3,1)$ $z_x = \frac{-3}{4}$ and $z_y = \frac{1}{4}$.

The chain rule is $dz = z_x\,dx + z_y\,dy$.

Plugging in $(1,3,1)$ we get $\boxed{(z-1) = -\frac{3}{4}(x-1) + \frac{1}{4}(y-3)}$

or $\frac{3}{4}x - \frac{1}{4}y + z = 1$

$\boxed{3x - y + 4z = 4}$

4. (**30 points**) Compute all the second partial derivatives of

$$f(x, y) = \sin(x)\cos(y) - \cos(x)\sin(y).$$

Observe that $f = \sin(x-y)$.

Thus $f_x = \cos(x-y)$, $f_y = -\cos(x-y)$.

So $f_{xx} = -\sin(x-y)$ $f_{yx} = \sin(x-y)$

$f_{xy} = \sin(x-y)$ $f_{yy} = -\sin(x-y)$.

5. (**15 points**) The surface of a mountain has equation $z = \ln(1 + 3x^2y^3)$. A bucket of water is emptied above the point $(2, 1)$. In which direction does the water flow? (You may give only the horizontal direction, as a cartographer would.)

We are looking for the unit vector u that minimizes $D_u(f)\big|_{(x,y)=(2,1)}$. This is (the unit vector of) $-\nabla f\big|_{(x,y)=(2,1)}$.

We compute $-\nabla f = -\langle z_x, z_y \rangle = -\left\langle \frac{6xy^3}{1+3x^2y^3}, \frac{9x^2y^2}{1+3x^2y^3} \right\rangle$,

so $-\nabla f\big|_{(x,y)=(2,1)} = \left\langle \frac{-12}{13}, \frac{-36}{13} \right\rangle$.

The water flows in the direction $\langle -1, -3 \rangle$.

6. (**15 points**) Find $\frac{dz}{dt}$, if $z = \sqrt{\left(\frac{\ln t}{7}\right)^4 + 3\left(\frac{\ln t}{7}\right)\left(\frac{t^3}{e^t}\right) + \left(\frac{t^3}{e^t}\right)^4 - 2}$.

Set $x = \frac{\ln t}{7}$, $y = \frac{t^3}{e^t}$.

Then $z = \sqrt{x^4 + 3xy + y^4 - 2}$, so $\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$

We compute $z_x = \frac{4x^3+3y}{2\sqrt{x^4+3xy+y^4-2}}$, $z_y = \frac{3x+4y^3}{2\sqrt{x^4+3xy+y^4-2}}$

$\frac{dx}{dt} = \frac{1}{7t}$, $\frac{dy}{dt} = \frac{3t^2e^t - t^3e^t}{e^{2t}}$, so

$$\boxed{\frac{dz}{dt} = \frac{4x^3+3y}{14t\sqrt{x^4+3xy+y^4-2}} + \frac{(3x+4y^3)(3t^2e^t - t^3e^t)}{2e^{2t}\sqrt{x^4+3xy+y^4-2}}, \quad \text{where } x = \frac{\ln t}{7},\ y = \frac{t^3}{e^t}}$$

7. (**30 points**) Consider the function $f(x,y) = 9xy - x^3 - y^3 - 6$. You do not need to compute the partial derivatives:

$$\begin{aligned} f_x &= 9y - 3x^2 \\ f_y &= 9x - 3y^2 \\ f_{xx} &= -6x \\ f_{xy} &= 9 \\ f_{yy} &= -6y. \end{aligned}$$

Determine whether or not the points below are critical points. If they are critical points, determine whether they are saddle points, local maxima, or local minima.

(a) $P = (0,0)$. $f_x = f_y = 0$, so it's critical.

$D = (0)(0) - 9^2 = -81 < 0$, so this is a saddle point.

(b) $Q = (0,3)$.

$f_x = 27 \neq 0$, so this isn't critical.

(c) $R = (3,0)$.

$f_x = -27 \neq 0$, so this isn't critical.

(d) $S = (3,3)$. $f_x = f_y = 0$, so this is a critical point.

$D = (-18)(-18) - 9^2 = 243 > 0$.

Since $f_{xx} < 0$, it's a local maximum.

8. (**Extra credit: 20 points**) Find the maximum and minimum values of the function $f(x, y) = x^2 + x^2y + y^3$ on the region $x^2 + y^2 \leq 1$. (This region has an interior.)

The critical points are where $f_x = 0, f_y = 0$

$$\left\{ \begin{array}{l} 2x + 2xy = 0 \\ x^2 + 3y^2 = 0 \end{array} \right\} \Rightarrow x = y = 0, \text{ so } (0,0) \text{ is a c.p.}$$

On the boundary $(x^2+y^2=1)$, we use Lagrange multipliers.

$\nabla f = \lambda \nabla g$ where $g = x^2 + y^2$ is the constraint.

$$\left\{ \begin{array}{ll} 2x + 2xy & = \lambda(2x) \\ x^2 + 3y^2 & = \lambda(2y) \\ x^2 + y^2 & = 1 \end{array} \right\}$$

$2x(1+y) = \lambda(2x)$ so $x = 0$ or $\lambda = 1+y$.

If $x = 0$, then $y = \pm 1$, so $(0,1)$ and $(0,-1)$ are candidates.

If $\lambda = 1+y$, then

$$\left\{ \begin{array}{ll} x^2 + 3y^2 & = (1+y)(2y) \\ x^2 + y^2 & = 1 \end{array} \right\} \Rightarrow 1 + 2y^2 = 2y + 2y^2$$

$1 = 2y$, so $y = \frac{1}{2}$,

So $x = \pm \frac{\sqrt{3}}{2}$.

So $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$ are candidates.

Candidate	$x^2 + x^2y + y^3$
$(0,0)$	0
$(0,1)$	1
$(0,-1)$	-1
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\frac{19}{8} = 2.375$
$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\frac{19}{8} = 2.375$

$$\boxed{\text{The minimum value is } -1 \text{ at } (0,-1) \text{ and the maximum is } \frac{19}{8} \text{ at } \left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right).}$$