Math 2163

Jeff Mermin's sections, Test 1, September 23

On the essay questions (# 4–11) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

Do not evaluate any integrals on this test. If you would take an integral, instead simplify the integrand and the limits of integration (if any), and leave the integral as your final answer.

1. (**30 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   In the statements below, x, y, z, and t are variables, a and b are numbers, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are vectors in $\mathbb{R}^3$, $C : \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a curve in space with associated frame $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$, and $f(x, y)$ is a function.

   (a) $\mathbf{x} \bullet (\mathbf{y} + \mathbf{z}) = \mathbf{x} \bullet \mathbf{y} + \mathbf{x} \bullet \mathbf{z}$.

   True.

   (b) $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.

   True.

   (c) $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.

   True.

   (d) $\mathbf{x} \bullet (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \bullet \mathbf{y})(\mathbf{x} \bullet \mathbf{z})$.

   False. (Try $x = y = z = \langle 1, 0, 0 \rangle$)

   (e) $|\mathbf{x} \bullet \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$.

   True. ($|x \cdot y| = |x|\,|y|\,|\cos\theta|$)

   (f) $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$.

   True.

   (g) $\mathbf{N} = \mathbf{B} \times \mathbf{T}$.

   True.

   (h) $\mathbf{x} - \mathbf{y} = \mathbf{y} - \mathbf{x}$.

   False.

   (i) $\frac{d\mathbf{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$.

   True.

   (j) The equations $x = 2$, $y = -1$, $z = 0$ define a line.

   False. (They define a point.)

2. (**20 points**) Let $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in $\mathbb{R}^3$. Are the following expressions vectors, scalars, or nonsense? (No justification is necessary on this problem, but wrong answers with good explanations may receive credit.)

   (a) $(((\underbrace{\mathbf{v}+\mathbf{w})+\mathbf{x}}_{}\,)\mathbf{y})+\mathbf{z}$

   $(u)y$ is nonsense.

   (b) $\mathbf{v}-(\mathbf{w}\times((\mathbf{x}+\mathbf{y})\times\mathbf{z}))$

   This is a vector.

   (c) $(\underbrace{\mathbf{v}-(\underbrace{\mathbf{w}(\underbrace{\mathbf{x}\bullet\mathbf{y}}_{a})}_{u})}_{u'})\bullet\mathbf{z}$

   $u' \bullet z$ is a scalar.

   (d) $(\underbrace{\mathbf{v}\bullet\mathbf{w}}_{a})(\underbrace{\mathbf{x}+(\mathbf{y}-\mathbf{z})}_{u})$ is a vector.

   (e) $(\underbrace{(\underbrace{\mathbf{v}+\mathbf{w}}_{u})\bullet\mathbf{x}}_{a})-(\underbrace{\mathbf{y}\bullet\mathbf{z}}_{b})$ is a scalar.

3. (**15 points**) Match the equations with the level curves shown below. (No justification is necessary on this problem, but wrong answers with good explanations may receive credit.)

   (a) $z = \sin(xy)$.

   (b) $z = \sin(x - y)$.

   (c) $z = (1 - x^2)(1 - y^2)$.

   (d) $z = e^x \cos y$.

4. (**25 points**) Let $\mathbf{x} = \langle -4, -3, 2\rangle$ and $\mathbf{y} = \langle -3, 5, -3\rangle$. Compute the following:

   (a) $5\mathbf{x} - 3\mathbf{y}$. $= 5\langle -4, -3, 2\rangle - 3\langle -3, 5, -3\rangle$

   $= \langle -20, -15, 10\rangle - \langle -9, 15, -9\rangle$

   $= \langle -20 - -9, -15 - 15, 10 - -9\rangle$

   $= \langle -11, -30, 19\rangle$.

   (b) $\mathbf{x} \bullet \mathbf{y}$.

   $= (-4)(-3) + (-3)(5) + (2)(-3)$

   $= 12 + -15 + -6$

   $= -9$.

   (c) $\mathbf{x} \times \mathbf{y}$.

   $= \langle (-3)(-3) - (2)(5), (2)(-3) - (-4)(-3), (-4)(5) - (-3)(-3)\rangle$

   $= \langle -1, -18, -29\rangle$

   (d) $(\mathbf{a} - \mathbf{b}) \times (\mathbf{b} - \mathbf{a})$. $= -(a-b)\times(a-b)$

   $= 0$.

5. (**5 points**) Find two points on the line $\langle x, y, z\rangle = (4, -5, 4) + \langle 2, 1, 4\rangle\, t$.

   Plugging in $t=0$, we get $(4, -5, 4)$.

   Plugging in $t=1$, we get $(6, -4, 8)$.

6. (**10 points**) Find three points on the plane $4x + y + 4z = -5$.

Plugging in $x = y = 0$, we get $4z = -5$, so $(0, 0, -5/4)$.

Plugging in $x = z = 0$ we get $y = -5$, so $(0, -5, 0)$.

Plugging in $x = z = 1$, we get $y + 8 = -5$, i.e. $y = -13$,

so $(1, -13, 1)$.

7. (**15 points**) Does the plane $F : x - 4y + 5z = 1$ contain $P = (-2, -5, -3)$? If not, find the distance from P to F, and give equations (in one of the standard forms) for the line perpendicular to F which passes through P.

At P, $x - 4y + 5z = -2 + 20 - 15 = 3 \neq 1$.

The distance is $\dfrac{|3-1|}{|\langle 1, -4, 5\rangle|} = \dfrac{2}{\sqrt{1^2 + 4^2 + 5^2}} = \dfrac{2}{\sqrt{42}}$.

The line in question is parallel to $n_F = \langle 1, -4, 5\rangle$ and passe through P.

So $(x, y, z) = (-2, -5, -3) + \langle 1, -4, 5\rangle t$.

8. (**15 points**) Let F be the plane consisting of all points equidistant from $(3, -4, -5)$ and $(5, 0, -3)$. Find an equation for F.

The normal vector is $\overrightarrow{PQ} = \langle 5-3, 0--4, -3--5 \rangle$
$= \langle 2, 4, 2 \rangle$.

The plane passes through the midpoint of P and Q, $(4, -2, -4)$.

Its equation is thus $2x + 4y + 2z = 2(4) + 4(-2) + 2(-4)$

$\boxed{2x + 4y + 2z = -8}$

9. (**15 points**) Let C be the topologist's screw $r(t) = (\sin \pi t, \cos \pi t, e^t)$. Find equations (in one of the standard forms) for the tangent line to C at the point $(0, -1, e)$.

First find t at $(0,-1,e)$: $\left\{ \begin{array}{l} \sin \pi t = 0 \\ \cos \pi t = -1 \\ e^t = e \end{array} \right\} \Rightarrow t = 1$.

Now the tangent line is parallel to $\left.\frac{dr}{dt}\right|_{t=1}$

$= \langle \pi \cos \pi t, -\pi \sin \pi t, e^t \rangle \big|_{t=1}$

$= \langle -\pi, 0, e \rangle$

so its equation is

$(x, y, z) = (0, -1, e) + \langle -\pi, 0, e \rangle (t-1)$.

10. (**Extra credit: 20 points**) Let C be the topologist's screw $r(t) = (\sin \pi t, \cos \pi t, e^t)$, and set $P = (0, -1, e)$ and $Q = (0, 1, 1)$. Choose and solve two of the three problems below. (Circle or otherwise clearly indicate your choices. If you attempt to choose more than two, I will grade none.)

   (a) Find the curvature of C at P.

   (b) Find $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at P.

   (c) Find the length of C between P and Q.

$t = 0$ at Q and $t = 1$ at P.

$$\frac{dr}{dt} = \langle \pi \cos \pi t, -\pi \sin \pi t, e^t \rangle \quad \text{and} \quad \frac{d^2r}{dt^2} = \langle -\pi^2 \sin \pi t, -\pi^2 \cos \pi t, e^t \rangle.$$

(a) $\kappa = \dfrac{\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right|}{\left|\langle \frac{dr}{dt} \rangle\right|^3}$. At $t = 1$, this is

$$\kappa = \frac{|\langle -\pi, 0, e\rangle \times \langle 0, -\pi^2, e\rangle|}{|\langle -\pi, 0, e\rangle|^3} = \frac{|\langle -e\pi^2, e\pi, \pi^3\rangle|}{|\langle -\pi, 9e\rangle|^3} = \frac{\sqrt{e^2\pi^4 + e^2\pi^2 + \pi^6}}{(\pi^2 + e^2)^{3/2}}$$

(b) T is the unit vector of $\frac{dr}{dt}$. At $t = 1$, this is $\dfrac{\langle -\pi, 0, e\rangle}{\sqrt{\pi^2 + e^2}}$.

B is the unit in the direction of $\frac{dr}{dt} \times \frac{d^2r}{dt^2}$. At $t = 1$, this is $\dfrac{\langle -e\pi^2, e\pi, \pi^3\rangle}{\sqrt{e^2\pi^4 + e^2\pi^2 + \pi^3}}$.

$$N = B \times T = \frac{1}{\sqrt{(\pi^2 + e^2)(e^2\pi^4 + e^2\pi^2 + \pi^3)}} \langle e^2\pi, -\pi^4 + e^2\pi^2, -e\pi^2 \rangle$$

(c) The arc length is $\displaystyle\int_{\text{start}}^{\text{end}} \left|\frac{dr}{dt}\right| dt = \int_{t=0}^{t=1} \sqrt{\pi^2 + e^2}\, dt$