Math 2163

Section 2, Final exam, December 11

On the essay questions (# 2–13) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

1. (**40 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   a, b, c, d, d' and L are numbers. $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^3$. R is a region in $\mathbb{R}^2$ or $\mathbb{R}^3$. F, f and g are smooth functions on their domains, which include R. $C : \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a parametric curve. (u, v) is an alternative coordinate system on $\mathbb{R}^2$.

   (a) $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$.

   False. $a \times b = -b \times a$.

   (b) If $\lim_{(x,y)\to(a,b)} f(x, y) = L,$, then $\lim_{(x,y)\to(a,b)} (\cos f(x, y)) = \cos L$.

   True.

   (c) If $R = \{(x, y, z) : x^2 + y^2 + z^2 \leq 9\}$, then $\iiint_R dV = 36\pi$.

   True.

   (d) If R_1 and R_2 are disjoint regions in $\mathbb{R}^3$, and R is their union, then $\iiint_R f(x, y, z)dV = \iiint_{R_1} f(x, y, z)dV + \iiint_{R_2} f(x, y, z)dV$.

   True.

   (e) $\iint_R f(x, y)g(x, y)dA = \left(\iint_R f(x, y)dA\right)\left(\iint_R g(x, y)dA\right)$.

   False.

   (f) If R is the sphere of radius one about the origin, then $\iiint_R f dV =$
   $$\int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=1} f\rho^2 \sin\phi \, d\rho d\phi d\theta.$$

   False. ϕ only goes from 0 to π.

   (g) There are eight possible orders of integration for a triple iterated integral.

   False. There are $3 \cdot 2 \cdot 1 = 6$.

   (h) The vector $\langle dx, dy, dz\rangle$ is normal to the graph of $z = f(x, y)$.

   False. It's a tangent vector.

(i) The distance between the planes $G : ax + by + cz = d$ and $G' : ax + by + cz = d'$ is $|d - d'|$.

False. It's $\dfrac{|d-d'|}{|\langle a,b,c\rangle|}$.

(j) $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 1.$

True. This is the Jacobian for changing from (x,y) to (u,v) and then back again, i.e., doing nothing.

(k) $\frac{d\mathbf{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$.

True.

(l) If (a, b) is a critical point of f, $f_{xx}(a, b) = 1$, and $f_{yy}(a, b) = -1$, then (a, b) is a saddle point of f.

True. $D = f_{xx}f_{yy} - (f_{xy})^2 \leq (1)(-1) < 0.$

(m) The volume of a right circular cylinder of radius 1 and height 2 is $\displaystyle\int_{z=0}^{z=2} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} dr d\theta dz.$

False. $dV = r\,dr\,d\theta\,dz$, not $dr\,d\theta\,dz$.

(n) Every trace of a hyperboloid is a hyperbola.

False. There are also ellipses.

(o) $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$.

True.

(p) If f has two local maxima, then it must have a local minimum.

False. consider $2x^2 - x^4 - y^2$, which has maxima at $(\pm 1, 0)$ and a saddle at $(0,0)$ but no minimum.

(q) $\frac{\partial}{\partial x}(f + g) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}$.

True.

(r) $\displaystyle\iint_R f\, dxdy = \iint_R f\, drd\theta.$

False. $dx\,dy = r\,dr\,d\theta$.

(s) $\nabla F(a, b, c)$ is normal to the surface $F(x, y, z) = 0$ at the point (a, b, c).

True. $\nabla F = \langle F_x, F_y, F_z \rangle$ and $\nabla F \bullet (dx, dy, dz) = F_x dx + F_y dy + F_z dz = dF = 0$.

(t) If $\lim_{x\to 0} f(x, b) = \lim_{y\to 0} f(a, y) = L$, then $\lim_{(x,y)\to(a,b)} f(x, y) = L$.

False. Consider $f = \dfrac{xy}{x^2+y^2}$ (and $\lim_{x\to 0} f(x,x)$).

2. (**10 points**) Let $\mathbf{a} = \langle 6, 10, -6\rangle$ and $\mathbf{b} = \langle 1, -2, 2\rangle$. Compute $\mathbf{a} \bullet \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$.

$a \bullet b = (6)(1) + (10)(-2) + (-6)(2) = -26$

$a \times b = \langle (10)(2) - (-6)(-2), (-6)(1) - (6)(2), (6)(-2) - (10)(1) \rangle$

$= \langle 8, -18, -22 \rangle$.

3. (**10 points**) Find an equation of the plane through the points $(4, -2, 1)$, $(4, -5, -3)$, and $(2, -4, -5)$.

Let P, Q, R be the points.

Two tangent vectors are $PQ = \langle 0, -3, -4 \rangle$

and $PR = \langle -2, -2, -6 \rangle$.

Their cross product $\langle 10, 8, -6 \rangle$ is normal.

So $\boxed{10x + 8y - 6z = 18}$ is an equation.

4. (**10 points**) Determine whether or not the planes $F : 5x - 2y + 2z = -3$ and $G : 5x - 2y + 2z = -5$ intersect. If they do, find an equation for the line of intersection. If they do not, find the distance between F and G.

The normals are $n_F = \langle 5, -2, 2 \rangle$ and $n_G = \langle 5, -2, 2 \rangle$.

These are the same, so the planes are parallel.

The distance is $\dfrac{|-3 - -5|}{|n_F|} = \dfrac{2}{\sqrt{33}}$.

5. (**10 points**) Find the directional derivative $D_{\mathbf{u}}(f)$ where $f(x,y) = \ln(2x^2 - y)$ and $\mathbf{u} = \langle \frac{5}{13}, \frac{12}{13} \rangle$.

We have $f_x = \dfrac{4x}{2x^2-y}$ and $f_y = \dfrac{-1}{2x^2-y}$.

Thus $D_u(f) = \frac{5}{13} f_x + \frac{12}{13} f_y$

$= \dfrac{20x - 12}{13(2x^2-y)}$.

6. (**10 points**) Find the equation for the tangent line to the surface C : $\mathbf{r}(t) = \langle \ln t, t^2, e^t \rangle$ at the point $(0, 1, e)$.

Solve for t: $\left\{ \begin{array}{l} \ln t = 0 \\ t^2 = 1 \\ e^t = e \end{array} \right\} \Rightarrow t = 1$.

Now $\frac{dr}{dt} = \langle \frac{1}{t}, 2t, e^t \rangle$, so $\left. \frac{dr}{dt} \right|_{t=1} = \langle 1, 2, e \rangle$.

The tangent line is

$(x,y,z) = (0,1,e) + \langle 1, 2, e \rangle s$.

(Apparently, $s = t-1$. Maybe $(x,y,z) = (0,1,e) + \langle 1,2,e \rangle (t-1)$ would be better.)

7. (**10 points**) Consider the function $f(x,y) = x^5 - 2y^3 + y - 5xy - 11$. You do not have to compute the derivatives

$$f_x = 5x^4 - 5y, \qquad f_y = 1 - 6y^2 - 5x$$
$$f_{xx} = 20x^3, \qquad f_{xy} = -5, \qquad f_{yy} = -12y$$

Determine whether the points below are critical points of f. If they are, classify them as local maxima, local minima, or saddle points.

(a) $P = (0,0)$. $f_y = 1$, so it's not critical.

(b) $Q = (-1,1)$. $f_x = f_y = 0$, so it's critical

$D = (-20)(-12) - (-5)^2 = 215 > 0$.

Since $f_{xx} < 0$, it's a local minimum.

8. (**10 points**) Compute $\displaystyle\int_{x=0}^{x=1} \int_{y=1}^{y=2} e^y \, dydx$.

$$= \int_{x=0}^{x=1} \Big[e^y\Big]_{y=1}^{y=2} dx$$

$$= \int_{x=0}^{x=1} (e^2 - e)\, dx$$

$$= \Big[(e^2 - e)x\Big]_{x=0}^{x=1} = \boxed{e^2 - e.}$$

9. (**10 points**) Write $\iint_R 12e^{x^2+y^2}\,dA$ as an iterated integral, where R is the region $R = \{x^2 + y^2 \leq 1,\, y \geq 0\}$. **Do not evaluate the integral.**

This is $0 \leq r \leq 1$, $0 \leq \theta \leq \pi$.

Our integral is $\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} 12e^{r^2}\, r\,dr\,d\theta$.

10. (**10 points**) Express $\int_C \langle x^2 + y^2, x^2 - y^2 \rangle \bullet d\mathbf{r}$ in a form that a Calculus II student would understand, where C is the semicircle $x = \sqrt{1-y^2}$ pointing from $(0, -1)$ to $(0, 1)$. (Calculus II students understand both numbers and simple definite integrals in one variable, so it is sufficient but **not necessary to evaluate the integral**).

The semicircle is $x^2+y^2=1$, i.e. the half of the unit circle $(\cos\theta, \sin\theta)$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Substituting throughout, we get

$$\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \langle \cos^2\theta + \sin^2\theta,\ \cos^2\theta - \sin^2\theta \rangle \bullet \langle -\sin\theta\, d\theta,\ \cos\theta\, d\theta \rangle$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} -\sin\theta\, d\theta + \cos^3\theta\, d\theta - \sin^2\theta\cos\theta\, d\theta$$

$$= \boxed{\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \left(-\sin\theta + \cos^3\theta - \sin^2\theta\cos\theta\right) d\theta}.$$

11. (**10 points**) Compute $\int_C \mathbf{F} \bullet d\mathbf{r}$, where C is the line from $(0,0)$ to $(2,1)$ followed by the line from $(2,1)$ to $(3,3)$, and $\mathbf{F} = \langle e^y, xe^y \rangle$.

$(e^y)_y = e^y$ and $(xe^y)_x = e^y$, so F is conservative and the path doesn't matter.

The potential function is $P(x,y) = \int e^y\,dx = \int xe^y\,dy$
$= xe^y + C_x = xe^y + C_y$.

So $P(x,y) = xe^y$ is good.

$$\int_C F \cdot dr = P(3,3) - P(0,0) = 3e^3 - 0e^0 = \boxed{3e^3}.$$

12. (**10 points**) Let $\mathbf{F} = \langle 2x + y, x + 2y \rangle$. Is $\mathbf{F}$ conservative? If it is, find a function $f(x,y)$ such that $\mathbf{F} = \nabla f$.

If F is conservative, then $F = \langle f_x, f_y \rangle$.

Then $f_{xy} = (2x+y)_y = 1$ and $(f_y)_x = (x+2y)_x = 1$.

So F is conservative.

The potential is $f = \int 2x+y\,dx = x^2 + xy + C_x$
and $f = \int x+2y\,dy = xy + y^2 + C_y$

i.e. $f = x^2 + xy + y^2 + C$.

$\boxed{f = x^2 + xy + y^2}$ will do.

13. (**Extra credit: 10 points**) Find the maximum and minimum values of the function $f(x, y) = (x-1)^2 + y^2$ on the circle $x^2 + y^2 \leq 4$.

f has critical points where $f_x = 0$, $f_y = 0$ i.e. $2(x-1) = 0$ and $2y = 0$, i.e. $(x,y) = (1,0)$.

This is inside the circle, so it's a candidate.

Meanwhile, we use Lagrange multipliers on the boundary.

$$\nabla f = \lambda \nabla g, \text{ so } \left\{ \begin{array}{l} 2(x-1) = \lambda(2x) \\ 2y = \lambda(2y) \\ x^2+y^2=4 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} y=0, x=2, \lambda=\frac{1}{2} \\ \text{or } y=0, x=-2, \lambda=\frac{-3}{2} \\ \text{or } \lambda=1, 2x-2=2x \text{ (FAIL!)} \end{array} \right\}$$

So the three candidates are $(1,0)$, $(2,0)$, and $(-2,0)$.

Point	$f(x,y)$
$(1,0)$	0
$(2,0)$	1
$(-2,0)$	9

The circle is closed and bounded, so maxima and minima exist.

Thus the $\boxed{\text{maximum is } 9}$ (at $(-2,0)$) and the $\boxed{\text{minimum is } 0}$ (at $(1,0)$).

14. (**Extra credit: 10 points**) In the space remaining on this page, write down a sequence which goes to infinity as fast as possible. ($\{a_n\}$ goes to infinity faster than $\{b_n\}$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$.) You may express the sequences in any way you'd like, including using words or recursion rather than explicit formulas (and in fact you may need to use some English to define any nonstandard notation.) However, it should be possible (at least in theory) for me to determine the exact value of every term in your sequence without any reference to the real world.