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Math 2163

Jeff Mermin's sections, Test 2, October 23

On the essay questions (# 2–7) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

1. (**30 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   In the statements below, a, b, c, d, d', and L are numbers, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors, $f(x,y)$ and $F(x,y,z)$ are smooth functions in the sense that all their partial derivatives are defined and continuous on their domains, and $C : \mathbf{r}(t)$ is a curve with associated vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$.

   (a) If (a,b) is a critical point of f, $f_{xx}(a,b) = 1$, and $f_{yy}(a,b) = -1$, then (a,b) is a saddle point of f.

   True. $D = (1)(-1) - (f_{xy})^2 \leq -1 < 0$.

   (b) If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, then $\lim\limits_{(x,y)\to(a,b)} (\cos f(x,y)) = \cos L$.

   True.

   (c) If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, then $\lim\limits_{x\to a} f(x,b) = L$.

   True.

   (d) The distance between the planes $F : ax + by + cz = d$ and $G : ax + by + cz = d'$ is $|d - d'|$.

   False. It's $\dfrac{|d-d'|}{|\langle a,b,c\rangle|}$.

   (e) $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$.

   True.

   (f) $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.

   True.

   (g) If z is defined implicitly as a function of x and y by $F(x,y,z) = 0$, then

   $$\frac{\partial z}{\partial x} = \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}.$$

   False. $0 = dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz$. Assuming $dy = 0$, we get $\frac{dz}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$.

   (h) There are functions $h_1(x)$ and $h_2(y)$ such that $f(x,y) = h_1(x) + h_2(y)$.

   False. Everything would be much easier if this were true.

   (i) $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$.

   False. $a \times b = -b \times a$.

   (j) There is a function $g(x,y)$ such that $g_x(x,y) = x^2 + y^2$ and $g_y(x,y) = x^2 - y^2$.

   False. If true, we'd have $g_{xy} = 2y$ and $g_{yx} = 2x$.

2. (**30 points**) Compute all the second partial derivatives of $f(x,y) = \frac{x^2-y^2}{xy}$.

First, notice $f = \frac{x}{y} - \frac{y}{x}$.

Now $f_x = \frac{1}{y} + \frac{y}{x^2}$ and $f_y = \frac{-x}{y^2} - \frac{1}{x}$.

Thus: $f_{xx} = \frac{-2y}{x^3}$, $f_{xy} = -\frac{1}{y^2} + \frac{1}{x^2}$; $f_{yx} = -\frac{1}{y^2} + \frac{1}{x^2}$, and $f_{yy} = \frac{2x}{y^3}$.

3. (**20 points**) Suppose that y is defined as a function of x by the relationship $x \sin y + y \cos x = 0$. Find $\frac{dy}{dx}$.

Let $F = x\sin y + y\cos x$ so $F = 0$.

Then $0 = dF = F_x\,dx + F_y\,dy$

$= (\sin y - y\sin x)\,dx + (x\cos y + \cos x)\,dy.$

Thus $(x\cos y + \cos x)\,dy = (\sin y - y\sin x)\,dx$

so $\frac{dy}{dx} = \frac{\sin y - y\sin x}{x\cos y + \cos x}$.

4. (**20 points**) Find an equation for the tangent plane to the hyperboloid $x^2 + y^2 - z^2 = 4$ at the point $(2, 1, 1)$.

Let $F = x^2+y^2-z^2$, so we want the tangent plane to $F=4$.

Now, along the surface $F=4$, we have

$dF = 0$

$F_x\,dx + F_y\,dy + F_z\,dz = 0$

$2x\,dx + 2y\,dy - 2z\,dz = 0.$

At $(2,1,1)$ this is $4\,dx + 2\,dy - 2\,dz = 0$,

i.e. $\boxed{4(x-2) + 2(y-1) - 2(z-1) = 0}$

or $\boxed{4x + 2y - 2z = 8}$

5. (**30 points**) Consider the function $f(x,y) = x^4 - 2x^2 + 3y^2$. You do not need to compute the partial derivatives:

$$
\begin{aligned}
f_x &= 4x^3 - 4x \\
f_y &= 6y \\
f_{xx} &= 12x^2 - 4 \\
f_{xy} &= 0 \\
f_{yy} &= 6.
\end{aligned}
$$

Determine whether or not the points below are critical points. If they are critical points, determine whether they are saddle points, local maxima, or local minima.

(a) $P = (0,0)$. $f_x = 0$ and $f_y = 0$, so it's critical.

$D = (-4)(6) - 0^2 = -24 < 0$, so this is a saddle point.

(b) $Q = (0,1)$.

$f_y = 6$, so this isn't critical.

(c) $R = (1,0)$.

$f_x = f_y = 0$, so this is critical.

$D = (8)(6) - 0^2 = 48 > 0$.

Since $f_{xx} > 0$, this is a local minimum.

6. (**20 points**) Using any appropriate method, find the absolute minimum value of the function $f(x, y) = x^2 + y^2$ subject to the constraint $xy = 3$. (You may assume such a minimum exists.)

We use Lagrange multipliers.

Setting $g = xy$ for the constraint, we know the minimum will satisfy $\nabla f = \lambda \nabla g$ for some λ.

$\nabla f = \langle 2x, 2y \rangle$ and $\nabla g = \langle y, x \rangle$.

Thus we want a simultaneous solution to

$$\begin{cases} 2x = \lambda y \\ 2y = \lambda x \\ xy = 3 \end{cases}$$

So $y = \frac{3}{x}$ and the first two equations are $\begin{cases} 2x = \frac{3\lambda}{x} \\ \frac{6}{x} = \lambda x \end{cases} \Rightarrow \begin{matrix} 2x^2 = 3\lambda \\ 6 = \lambda x^2 \end{matrix} \Rightarrow x^2 = \pm 3,\ \lambda = \pm 2.$

(But $x^2 \geq 0$, so $x^2 = 3$, so $x = \pm\sqrt{3}$ and the candidates are $P_1 = (\sqrt{3}, \sqrt{3})$, $P_2 = (-\sqrt{3}, -\sqrt{3})$.

We have $f(P_1) = 6$ and $f(P_2) = 6$. Thus $\boxed{\text{The minimum is } 6.}$

(Because the constraint $xy = 3$ doesn't describe a closed and bounded region, we can't know whether these critical points are actually minima without some other technique. Thankfully the problem told us that a minimum does exist. (But if it had asked instead for a maximum, there'd be no answer.))

7. (**Extra credit: 20 points**) Name, and write down possible equations for, each of the surfaces pictured below.

(a)

(b)

(c)

(d)

(e)