Math 2163

Jeff Mermin's sections, Test 1, September 18

On the essay questions (# 2–8) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

Do not evaluate any integrals on this test. If you would take an integral, instead simplify the integrand and the limits of integration (if any), and leave the integral as your final answer.

1. (**30 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   In the statements below, x, y, z, and t are variables, a, b, c, d, d', and L are numbers, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors, $C : \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a curve in space with associated $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, and κ, and $f(x, y)$ is a function.

   (a) $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{b})(\mathbf{a} \bullet \mathbf{c})$.

   False. (Try $a = b = c = \langle 1, 0, 0\rangle$.)

   (b) $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$.

   True.

   (c) $\frac{d\mathbf{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$.

   True.

   (d) There are functions $g(x)$ and $h(y)$ such that $f(x, y) = g(x) + h(y)$.

   False. (If this were true there'd be no need for any of the content of the class. We'd just focus on finding g, h.)

   (e) $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.

   True.

   (f) The distance between the planes $F : ax + by + cz = d$ and $G : ax + by + cz = d'$ is $|d - d'|$.

   False. It's $\frac{d-d'}{|\langle a,b,c\rangle|}$.

   (g) If C is a circle, then κ is its radius.

   False. ($\frac{1}{\kappa}$ is the radius.)

   (h) $|\mathbf{a} \bullet \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$.

   True. $|a \cdot b| = |a||b||\cos\theta|$, and $|\cos\theta| \leq 1$.

   (i) $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$.

   False. $a \times b = -b \times a$.

   (j) If $\lim_{(x,y)\to(a,b)} f(x, y) = L$, then $\lim_{x\to a} f(x, b) = L$.

   True.

2. (**30 points**) Let $\mathbf{a} = \langle 7, 7, 7\rangle$ and $\mathbf{b} = \langle 8, 9, -10\rangle$. Compute the following:

(a) $10\mathbf{a} - 10\mathbf{b}$.

$= 10\langle 7, 7, 7\rangle - 10\langle 8, 9, -10\rangle$

$= \langle 70, 70, 70\rangle - \langle 80, 90, -100\rangle$

$= \langle 70-80, 70-90, 70+100\rangle$

$= \boxed{\langle -10, -20, 170\rangle}$

(b) $\mathbf{a} \bullet \mathbf{b}$.

$= 7\cdot 8 + 7\cdot 9 + 7\cdot -10$

$= 56 + 63 - 70$

$= \boxed{49.}$

(c) $\mathbf{a} \times \mathbf{b}$. $= \langle 7(-10) - 7(9), 7(8) - 7(-10), 7(9) - 7(8)\rangle$

$= \boxed{\langle -133, 126, 7\rangle}$

(d) $(\mathbf{a} - \mathbf{b}) \times \mathbf{b}$. $= a \times b - b \times b$

$= a \times b - 0$

$= a \times b$

$= \boxed{\langle -133, 126, 7\rangle}$

3. (**15 points each**) Show that the limits do not exist, or give strong evidence that they do.

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$.

Approaching $(0,0)$ along the x-axis $(y=0)$

We get $\lim_{x\to 0} \frac{x^2-0}{x^2+0} = 1$.

Approaching along the y-axis $(x=0)$,

we get $\lim_{y\to 0} \frac{0-y^2}{0+y^2} = -1$.

Approaching along the line $y=x$, we get

$\lim_{x\to 0} \frac{x^2-x^2}{x^2+x^2} = \lim_{x\to 0} \frac{0}{2x^2} = 0$.

These don't agree, so the limit doesn't exist.

(b) $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+2y^2}$.

Approaching along $y=mx$ (for any slope m)

we get $\lim_{x\to 0} \frac{3m\,x^3}{x^2+2m^2x^2} = \lim_{x\to 0} \frac{3m\,x^3}{(2m^2+1)\,x^2} = \frac{3m}{2m^2+1} \lim_{x\to 0} x$

$= 0.$

These all agree, so it's likely the limit is equal to 0.

Approaching along $y=x^t$ (for any positive exponent t)

we get $\lim_{x\to 0} \frac{3x^{2+t}}{x^2+2x^{2t}} = \begin{cases} \lim_{x\to 0} \frac{3x^{2+t}}{x^2} = 0 & (\text{if } 2\leq 2t) \\ \lim_{x\to 0} \frac{3x^{2+t}}{2x^{2t}} < \frac{3x^{2t}}{2x^{2t}} = 0 & (\text{if } 2>2t) \end{cases}$

These still agree, which is stronger evidence.

We'll learn how to prove this limit is actually zero in MATH 4023.

4. (**20 points**) Find the equation of the plane containing the points $(-10, 10, -6)$, $(2, 0, 10)$, and $(-9, -5, -3)$.

Call $P = (-10, 10, -6)$, $Q = (2, 0, 10)$, and $R = (-9, -5, -3)$.

Two vectors in our plane are $\vec{PQ} = \langle 12, -10, 16 \rangle$
and $\vec{QR} = \langle -11, -5, -13 \rangle$.

Thus a normal vector is $n = \vec{PQ} \times \vec{QR} = \langle 130 + 80, -176 + 156, -60 - 110 \rangle$
$= \langle 210, -20, -170 \rangle$;

we scale that down to $\langle 21, -2, -17 \rangle$.

The equation has the form $21x - 2y - 17z = D$.
To find D, we plug in P: $21(-10) - 2(10) - 17(-6)$
$= -210 - 20 + 102 = -128$.

So the equation is $\boxed{21x - 2y - 17z = -128}$.

5. (**15 points**) Determine whether the planes $F : 2x + -6y - 7z = -6$ and $G : 7x + 5y = 5$ intersect. If they do, find equations for the line of intersection. If they do not, find the distance between F and G.

The normal vectors are $n_F = \langle 2, -6, -7 \rangle$
and $n_G = \langle 7, 5, 0 \rangle$. These aren't parallel, so the planes intersect.

To find the line of intersection, we need either two points or a point and a direction.

To find a point, invent an extra equation (e.g. $x = 0$)
and solve
$$\begin{cases} F: 2x - 6y - 7z = -6 \\ G: 7x + 5y = 5 \\ x = 0 \end{cases}$$
; we get $(0, 1, 0)$.

To find a direction, observe that $n_F \times n_G = \langle 35, -49, 52 \rangle$
is perpendicular to both normals, so parallel to both planes
(i.e., is the direction of the line.)

Thus the line of intersection is parametrized by
$\boxed{\langle x, y, z \rangle = (0, 1, 0) + \langle 35, -49, 52 \rangle t}$

6. (**15 points**) Find the length of the curve $C : \mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ on the interval $1 \leq t \leq 4$.

This is $\displaystyle\int_{\text{start}}^{\text{end}} |ds| = \int_{\text{start}}^{\text{end}} \left|\frac{ds}{dt}\right| dt$

$$= \int_{t=1}^{t=4} \sqrt{\left(\frac{ds}{dt}\right)^2}\, dt = \int_{t=1}^{t=4} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_{t=1}^{t=4} \sqrt{(1)^2 + (2t)^2 + (3t^2)^2}\, dt$$

$$= \boxed{\int_{t=1}^{t=4} \sqrt{1 + 4t^2 + 9t^4}\, dt}$$

(Read the directions so you know when to stop.)

7. (**10 points**) Find equations for the tangent line to the curve $C : \mathbf{r}(t) = \left\langle t^2, \frac{t}{t+1}, e^{2t} \right\rangle$ at the point $(0, 0, 1)$.

C passes throug $(0,0,1)$ when $\left\{\begin{array}{l} t^2 = 0 \\ \frac{t}{t+1} = 0 \\ e^{2t} = 1 \end{array}\right\}$, i.e., $t=0$.

The tangent vector is $\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$

$= \left\langle 2t, \frac{1}{(t+1)^2}, 2e^{2t} \right\rangle$.

At $t=0$, this is $\langle 0, 1, 2 \rangle$.

Thus our line is parametrized by

$$\boxed{(x,y,z) = (0,0,1) + \langle 0,1,2 \rangle t}.$$

8. (**Extra credit: 20 points**) Consider the curve $C : \mathbf{r}(t) = \langle e^t \sin t, t^2, e^t \cos t \rangle$. Compute $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at the point $(0, 0, 1)$, and find an equation for the osculating plane of C at this point.

We compute $\left\{ \begin{array}{l} e^t \sin t = 0 \\ t^2 = 0 \\ e^t \cos t = 1 \end{array} \right\} \Rightarrow t = 0.$

T is the unit vector of $\frac{dr}{dt}$ at $t = 0$;

$$\frac{dr}{dt} = \langle e^t \sin t + e^t \cos t, 2t, e^t \cos t - e^t \sin t \rangle,$$

so $\left.\frac{dr}{dt}\right|_{t=0} = \langle 1, 0, 1 \rangle$, and $\left.T\right|_{t=0} = \frac{\langle 1, 0, 1 \rangle}{\sqrt{2}}$

N is the (perpendicular to T part of) $\frac{d^2r}{dt^2} = \langle 2e^t \cos t, 2, -2e^t \sin t \rangle$;

at $t = 0$, $\frac{d^2 r}{dt^2} = \langle 2, 2, 0 \rangle$.

B is the unit vector of $T \times \frac{d^2r}{dt^2} = \langle -2, 2, 2 \rangle$.

Thus $B = \frac{1}{\sqrt{3}} \langle -1, 1, 1 \rangle$

Now $N = B \times T = \frac{1}{\sqrt{6}} \langle 1, -2, -1 \rangle$.

Meanwhile, B is normal to the osculating plane,

so its equation is $x - 2y - z = [0 - 2(0) - (1)]$

i.e. $\boxed{x - 2y - z = -1}$