Math 2163

Jeff Mermin's section, Quiz 7, November 16

1. **(7 points each)** Express the following as iterated integrals. You may use any coordinate system you like, as long as you use it correctly and justify your work.

   (a) The volume of the region R using an iterated integral (or integrals), if R is the region between the surfaces $z = x^2 + y^2$ and $z = 25 - x^2 - y^2$, and above the first and fourth quadrants (that is, $x \geq 0$).

The equations appear to be

A $z \geq x^2+y^2$

B $z \leq 25-x^2-y^2$

C $x \geq 0$.

From A and B, we get

D $x^2+y^2 \leq 25-x^2-y^2$

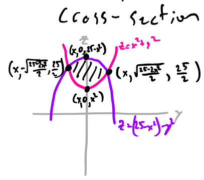
$x^2+y^2 \leq \frac{25}{2}$.

Putting x outside:

We have $0 \leq x \leq \frac{25}{2}$, (C) (D)

so $\displaystyle\int_{x=0}^{x=\frac{25}{2}} \left(\iint_{D_x} dy\,dz \right) dx$

where D_x is the cross-section



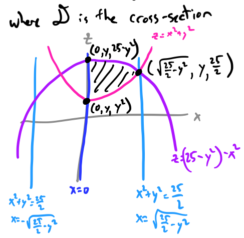
We get $\displaystyle\int_{x=0}^{x=\frac{25}{2}} \int_{y=-\sqrt{\frac{25-2x^2}{2}}}^{y=\sqrt{\frac{25-2x^2}{2}}} \int_{z=x^2}^{z=25-x^2-y^2} 1\,dz\,dy\,dy$

Putting y outside

From D, $-\frac{5}{\sqrt{2}} \leq y \leq \frac{5}{\sqrt{2}}$, so

the volume is $\displaystyle\int_{y=-\frac{5}{\sqrt{2}}}^{y=\frac{5}{\sqrt{2}}} \left(\iint_{D} 1\,dx\,dz \right) dy$

where D is the cross-section



So our volume is

$\displaystyle\int_{y=-\frac{5}{2}}^{y=\frac{5}{2}} \int_{x=0}^{x=\sqrt{\frac{25}{2}-y^2}} \int_{z=x^2+y^2}^{z=25-y^2-x^2} 1\,dz\,dx\,dy$

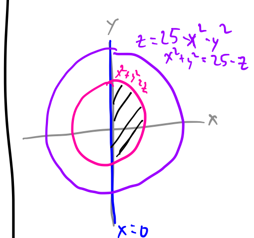
Putting z outside:

From A $z \geq 0$, and from B $z \leq 25$,

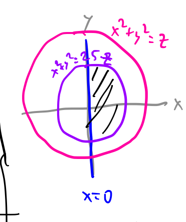
so we have

$\displaystyle\int_{z=0}^{z=25} \left(\iint_{D} 1\,dx\,dy \right) dz$,

where D is the cross-section



(if $z < \frac{25}{2}$)



(if $z > \frac{25}{2}$).

This is super annoying, since the curves are differently configured for different z. So we do one of the others, instead of dealing with

$\displaystyle\int_{z=0}^{z=\frac{25}{2}} \int_{x=0}^{x=\sqrt{z}} \int_{y=-\sqrt{z-x^2}}^{y=\sqrt{z-x^2}} 1\,dy\,dx\,dz$

$\displaystyle+ \int_{z=\frac{25}{2}}^{z=25} \int_{x=0}^{x=\sqrt{25-z}} \int_{y=-\sqrt{25-z-x^2}}^{y=\sqrt{25-z-x^2}} 1\,dy\,dx\,dz$

We can buy some simplicity using polar coordinates.

$x^2+y^2=z$ is $r^2=z$, $z=25-x^2-y^2$ is $z=25-r^2$, and $x \geq 0$ is $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

The area becomes

$$\int_{r=0}^{r=\sqrt{\frac{25}{2}}} \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{z=r^2}^{z=25-r^2} 1\,dz\,dy\,dx = \int_{r=0}^{r=\sqrt{\frac{25}{2}}} \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{z=r^2}^{z=25-r^2} r\,dz\,d\theta\,dr$$

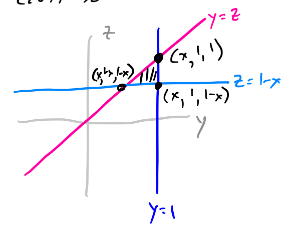
(since $dy\,dx = r\,dr\,d\theta$).

(b) The mass of the tetrahedron with vertices $(1,\overset{P}{1},1)$, $(1,\overset{Q}{0},0)$, $(\overset{R}{0},1,1)$, and $\underset{S}{(1,1,0)}$, if its density is given by $\rho(x,y,z) = x+y$.

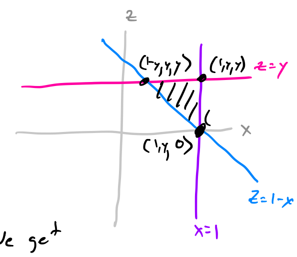
The equations of the planes through 3 vertices are:

PQR : $y-z=0$ Each variable goes from 0 to 1.
PQS : $x=1$
PRS : $y=1$
QRS : $x+z=1$

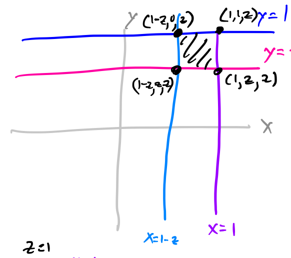
A $\underline{x \text{ outside}}$ cross-section looks like



We get
$$\int_{x=0}^{x=1}\int_{y=1-x}^{y=1}\int_{z=1-x}^{z=y} x+y \, dz\,dy\,dx \quad \text{or} \quad \int_{x=0}^{x=1}\int_{z=1-x}^{z=1}\int_{y=z}^{y=1} x+y \, dy\,dz\,dx$$



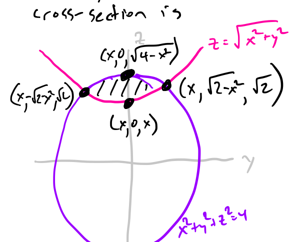
We get
$$\int_{y=0}^{y=1}\int_{x=1-y}^{x=1}\int_{z=1-x}^{z=y} x+y \, dz\,dx\,dy \quad \text{or} \quad \int_{y=0}^{y=1}\int_{z=0}^{z=y}\int_{x=1-z}^{x=1} x+y \, dx\,dz\,dy$$



$$\int_{z=0}^{z=1}\int_{x=1-z}^{x=1}\int_{y=z}^{y=1} x+y \, dy\,dx\,dz \quad \text{or} \quad \int_{z=0}^{z=1}\int_{y=z}^{y=1}\int_{x=1-z}^{x=1} x+y \, dx\,dy\,dz$$

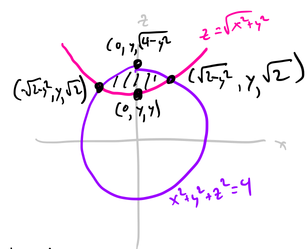
(c) The volume of the "ice cream cone" above $z=\sqrt{x^2+y^2}$ and inside $x^2+y^2+z^2=4$. The equations are $z=\sqrt{x^2+y^2}$ and $x^2+y^2+z^2=4$. They meet at the circle $\{z=\sqrt{2};\ x^2+y^2=2\}$. We have $-\sqrt{2}<x<\sqrt{2}$; $-\sqrt{2}<y<\sqrt{2}$, $0<z<2$.

If we put x outside, the cross-section is



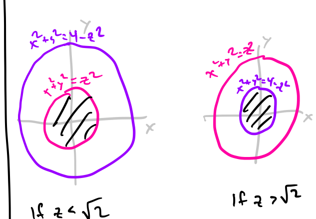
We get $\displaystyle\int_{x=0}^{x=1}\int_{y=-\sqrt{2-x^2}}^{y=\sqrt{2-x^2}}\int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{4-x^2-y^2}} dz\,dy\,dy$

If we put y outside, the cross-section is



We get
$$\int_{y=0}^{y=1}\int_{x=-\sqrt{2-y^2}}^{x=\sqrt{2-y^2}}\int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{4-x^2-y^2}} dz\,dy\,dx$$

If we put z outside, the cross-section is



We don't want to deal with that.

In polar coordinates, the equations are $z=r$ and $r^2+z^2=4$ (and $0<\theta<2\pi$). They meet at $r=z=2$.

We get
$$\int_{r=0}^{r=2}\int_{\theta=0}^{\theta=2\pi}\int_{z=r}^{z=\sqrt{4-r^2}} r\,dz\,d\theta\,dr$$

(because $dx\,dy = r\,dr\,d\theta$).

In spherical coordinates, the equations are $\phi=\frac{\pi}{4}$ and $\rho=2$.

(and $\phi>0$, $\rho>0$, $0<\theta<2\pi$).

We get
$$\int_{\rho=0}^{\rho=2}\int_{\phi=0}^{\phi=\frac{\pi}{4}}\int_{\theta=0}^{\theta=2\pi} \rho^2\sin\phi \, d\theta\,d\phi\,d\rho$$

(because $dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$).