Math 2163

Jeff Mermin's section, Quiz 5, October 20

1. **(2 points each)** Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false" or "possibly false".) No justification is necessary. Write out the whole word "true" or "false".

   On these problems, a, b, and c are numbers, x, y, and z are the usual rectangular coordinates, $f = f(x, y)$ and $F = F(x, y, z)$ are smoothly differentiable multivariable functions, and P is a point.

   (a) $\nabla f(a, b)$ is normal to the level curve of $f(x, y)$ passing through (a, b).

   This is $\boxed{\text{true.}}$

   (b) If D is neither closed nor bounded, then f fails to have a global maximum on D.

   This is false. perhaps $D = \mathbb{R}$ and $f(x,y) = -x^2 - y^2$.

   (c) If $\mathbf{u}$ is the unit vector in the direction of ∇F at P, then $D_{\mathbf{u}}(F)(P) = \|\nabla F\|$.

   This is $\boxed{\text{true.}}$ We have $u = \frac{\nabla F}{\|\nabla F\|}$, and $D_u(F) = u \cdot \nabla F$

   $= \frac{\nabla F \cdot \nabla F}{\|\nabla F\|} = \frac{\|\nabla F\|^2}{\|\nabla F\|}$.

   (d) $\nabla F(a, b, c)$ is tangent to the surface $F(x, y, z) = 0$ at the point (a, b, c).

   This is $\boxed{\text{false}}$. ∇F is normal to the level surfaces of F.

   (e) If f has two local maxima, then it must have a local minimum.

   This is $\boxed{\text{false}}$ For example, if $f(x,y) = \cos x - y^2$, there are infinitely many maxima (and saddle points) but no minima.

2. **(4 points)** Find the equation for the tangent plane to the level surface of $F(x, y, z) = x^2 + y^2 + z^2 - 3xyz$ at the point $P = (2, 5, 7)$.

We have $dF = F_x\,dx + F_y\,dy + F_z\,dz$.

We compute $F_x = 2x - 3yz$ and so, at $(2,5,7)$ we have

$F_y = 2y - 3xz$ $\quad F_x = -101$, $\quad dx = x - 2$

$F_z = 2z - 3xy$ $\quad F_y = -36$ $\quad dy = y - 5$

$F_z = -16$ $\quad dz = z - 7$

Finally, $dF = 0$ since F is constant along the level surface.

Thus $dF = F_x\,dx + F_y\,dy + F_z\,dz$

becomes $\boxed{0 = -101(x-2) - 36(y-5) - 16(z-7)}$.

3. **(4 points)** Consider the function $f(x, y) = x^3 + 9xy - 9y^2 - 21x$. You may assume that

$$f_x = 3x^2 + 9y - 21 \qquad f_y = 9x - 18y$$

$$f_{xx} = 6x \qquad f_{xy} = 9 \qquad f_{yy} = -18$$

Decide whether the points below are critical points of f. Then, if they are, classify them as local maxima, local minima, or saddle points.

(a) $P = (0, 0)$.

We get $f_x = -21 \neq 0$, so it's $\boxed{\text{not a critical point.}}$

(b) $Q = (1, 2)$.

$f_y = -27 \neq 0$, so it's $\boxed{\text{not a critical point.}}$

(c) $S = (2, 1)$. $f_x = 0$ and $f_y = 0$, so it's a critical point.

We compute $D = f_{xx} f_{yy} - (f_{xy})^2$
$= (12)(-18) - 81 < 0$, so it's a $\boxed{\text{saddle point.}}$

4. **(2 points)** Does the function f above have any other critical points? Find them, or verify that they are all listed above.

$$\frac{-9 \pm \sqrt{33}}{24}$$

We solve $f_x = f_y = 0$: $3x^2 + 9y - 21 = 0$ (A)
$9x - 18y = 0$ (B).

From (B), we get (C) $x = 2y$. Substituting (C) in (A), we get $12y^2 + 9y - 21 = 0$, which has two solutions: $y = 1$ and $y = \frac{-7}{4}$. So $\boxed{\left(\frac{-7}{2}, \frac{-7}{4}\right)}$ is critical.

Extra Credit **(2 points)**: Sketch the contour map of the function f above. (This will probably require checking the type(s) of any critical points you found in problem 4.)