Math 2163

Jeff Mermin's section, Quiz 4, October 6

1. **(2 points each)** Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false" or "possibly false".) No justification is necessary. Write out the whole word "true" or "false".

   x and y are the usual rectangular coordinates, $f = f(x, y)$ and $g = g(x, y)$ are smoothly differentiable multivariable functions, and a, b, and L are numbers.

   (a) $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$. This is

   $(f_y)_x = (f_x)_y$, i.e. $f_{yx} = f_{xy}$. It's true.

   (b) If $\lim_{(x,y)\to(a,b)} f(x, y) = L$, then $\lim_{x\to a} f(x, b) = L$.

   Since $\lim_{(x,y)\to(a,b)} f = L$, we have $f(x^*, y^*) \approx L$ whenever $x^* \approx a$, $y^* \approx b$.
   Thus $f(x^*, b) \approx L$ whenever $x^* \approx a$, so the statement is true.

   (c) For every f, there are functions $h_1(x)$ and $h_2(y)$ such that $f(x, y) = h_1(x) + h_2(y)$.

   This is false. Consider $f(x,y) = xy$.
   We have $f_{xy} = 1$, but $(h_1 + h_2)_{xy} = (h_1)_{yx} + (h_2)_{xy} = 0 + 0 \neq 1$.

   (d) $\frac{\partial}{\partial x}(fg) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial x}$.

   This is false. The product rule is $(fg)_x = f g_x + f_x g$.

   (e) There is a smoothly differentiable function $g(x, y)$ such that $g_x(x, y) = x^2 + y^2$ and $g_y(x, y) = x^2 - y^2$.

   This is false. We get $g_{xy} = (x^2+y^2)_y = 2y$, but $g_{yx} = (x^2-y^2)_x = 2x$.

2. **(4 points)** Find the equation for the tangent plane to the graph of $z = x^3 - xy^2$ at the point $(2, 3, f(2, 3))$.

   We have $dz = z_x\,dx + z_y\,dy$.

   Computing, $z_x = 3x^2 - y^2$ and $z_y = -2xy$.

   Near $(2,3)$, we have:
   - $z = 2^3 - 2(3^2) = -10$
   - $z_x = 3(2^2) - 3^2 = 3$
   - $z_y = -2(2)(3) = -12$
   - $dz = z - 10$
   - $dx = x - 2$
   - $dy = y - 3$

   Substituting everything the tangent plane is

   $$\boxed{(z - 10) = 3(x-2) - 12(y-3)}.$$

3. **(2 points)** State the chain rule (the one I've been writing on the board all week, not the one in the textbook).

If z is a function of (x,y), then

$$\boxed{dz = z_x\,dx + z_y\,dy}$$

4. **(4 points)** Find $\frac{dz}{dt}$, if

$$z(t) = \left(\ln\left(\frac{t^4+1}{t^2+3}\right)\right)^3 (t\sin(t^2)) + \left(\ln\left(\frac{t^4+1}{t^2+3}\right)\right)^2 (t\sin(t^2))^2 - (t\sin(t^2))^5.$$

(You will probably want to create some new notation, and express your answer in terms of this notation.)

Set $x = \ln\left(\frac{t^4+1}{t^2+3}\right) = \ln(t^4+1) - \ln(t^2+3)$

and $y = t\sin(t^2)$.

Then $z = x^3 y + x^2y^2 - y^5$.

So $dz = z_x\,dx + z_y\,dy$; and $\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$.

We compute: $z_x = 3x^2y + 2xy^2$, $z_y = x^3 + 2x^2y - 5y^4$

$\frac{dx}{dt} = \frac{4t^3}{t^4+1} - \frac{2t}{t^2+3}$, $\frac{dy}{dt} = \sin(t^2) + t\cdot\cos(t^2)\cdot 2t$

$= \sin(t^2) + 2t^2\cos t^2$.

Thus $\frac{dz}{dt} = (3x^2y + 2xy^2)\left(\frac{4t^3}{t^4+1} - \frac{2t}{t^2+3}\right) + (x^3 + 2x^2y - 5y^4)(\sin(t^2) + 2t^2\cos t^2)$

where $x = \ln\left(\frac{t^4+1}{t^2+3}\right)$ and $y = t\sin t^2$

Extra Credit **(3 points)**: Explain the joke (both the setup and the punchline):

> e^x and a constant function are out for a walk together when they see a differential operator approaching. The constant function panics and starts to run away, but e^x stops him and asks what's wrong. The constant function, terrified, exclaims "That's a differential operator, it'll annihilate me!"
>
> "Nonsense," replies e^x, "differential operators aren't dangerous. Watch, I'll walk right up to him and have a nice conversation and nothing will happen." So e^x walks over to the differential operator. "Hi there," he says, "I'm e^x."
>
> "Hi," says the operator, "I'm $\frac{\partial}{\partial y}$."

3 points for addressing 3 questions:

- Why is the constant afraid?
- Why is e^x not afraid?
- What's the joke?