Math 2163

Jeff Mermin's section, Quiz 2, September 1

1. **(2 points each)** Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false" or "possibly false".) No justification is necessary. Write out the whole word "true" or "false".

   On these problems, $\mathbf{r} = \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ and $\mathbf{s} = \mathbf{s}(t)$ are parametric curves in $\mathbb{R}^3$, t is a parameter, and $\mathbf{v}$ and $\mathbf{w}$ are (nonzero) vectors in $\mathbb{R}^3$.

   (a) $\frac{d}{dt}(\mathbf{r} \bullet \mathbf{s}) = \left(\frac{d\mathbf{r}}{dt} \bullet \mathbf{s}\right) + \left(\mathbf{r} \bullet \frac{d\mathbf{s}}{dt}\right)$.

   This is $\boxed{\text{true.}}$ It's the product rule.

   (b) $\frac{d}{dt}(\mathbf{r} \bullet \mathbf{s}) = \frac{d\mathbf{r}}{dt} \bullet \frac{d\mathbf{s}}{dt}$.

   This is $\boxed{\text{false.}}$ The correct product rule is $\frac{d}{dt}(r \bullet s) = \frac{dr}{dt} \bullet s + r \bullet \frac{ds}{dt}$.

   (c) $\frac{d}{dt}(\mathbf{r} \times \mathbf{s}) = \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{s}}{dt}$.

   This is $\boxed{\text{false}}$. The correct product rule is $\frac{d}{dt}(r \times s) = \frac{dr}{dt} \times s + r \times \frac{ds}{dt}$.

   (d) $\int_{t=0}^{t=1} \frac{d\mathbf{r}}{dt} dt = \mathbf{r}(1) - \mathbf{r}(0)$.

   This is $\boxed{\text{true}}$. It's the fundamental theorem of calculus.

   (e) $\mathbf{v}$ and $\mathbf{w}$ are perpendicular if and only if $\mathbf{v} \times \mathbf{w} = \mathbf{0}$.

   This is false. $v \times w = 0$ means v and w are parallel;
   $v \bullet w = 0$ means they're perpendicular.

2. **(5 points)** Find the point of intersection between the line $\ell : (x, y, z) = (-1, -3, 5) + \langle -2, 2, 0 \rangle t$ and the plane $L : 3x - y + 3z = -25$, or explain why no such point exists.

   The point (if it exists) is $P = (x, y, z)$ making it so

   - $3x - y + 3z = -25$ (i.e, P is on L)
   - $\left\{\begin{array}{l} x = -1-2t \\ y = -3+2t \\ z = 5 \end{array}\right\}$ for some t (i.e., P is on ℓ)

   Solving these simultaneously, we get $3(-1-2t) - (-3+2t) + 3(5) = -25$

   i.e., $t = 5$.

   Plugging in $t = 5$ to the equations for ℓ,

   we get $\boxed{(x, y, z) = (-11, 7, 5)}$.

3. **(5 points)** Consider the semidouble quasihelix C parametrized by

$$C : \mathbf{r}(t) = (\cos \pi t, \sin \pi t, t^2)$$

and the point $P = \left(0, 1, \dfrac{1}{4}\right)$.

(a) Show that P is on C. We need to find a t so that $r(t) = (0, 1, \frac{1}{4})$, i.e. a simultaneous solution to

$$\left\{ \begin{array}{l} \cos \pi t = 0 \\ \sin \pi t = 1 \\ t^2 = 1/4 \end{array} \right\}.$$

$\boxed{t = \frac{1}{2}}$ works.

(b) Find the equation for the tangent line to C at P.

At an arbitrary t, the tangent direction is $\frac{dr}{dt} = \langle -\pi \sin \pi t, \pi \cos \pi t, 2t \rangle$.

At P, $t = \frac{1}{2}$, so the tangent direction is $\left.\frac{dr}{dt}\right|_{t=\frac{1}{2}} = \langle -\pi \sin \frac{\pi}{2}, \pi \cos \frac{\pi}{2}, 2(\frac{1}{2}) \rangle$

$= \langle -\pi, 0, 1 \rangle$

We want the line through P with this direction vector, i.e.,

$\ell: \boxed{(x, y, z) = (0, 1, \frac{1}{4}) + \langle -\pi, 0, 1 \rangle s.}$

(or, taking $s = t - \frac{1}{2}$ so we hit P at $t = \frac{1}{2}$, $\boxed{(x, y, z) = (0, 1, \frac{1}{4}) + \langle -\pi, 0, 1 \rangle (t - \frac{1}{2})}$.

Extra Credit **(3 points)**: An object in space experiences an acceleration given by the function $\mathbf{a}(t) = \langle t^2, 2t, 1 - t^3 \rangle$. It passes through the points $(0, 0, 0)$ and $(1, 2, 3)$ at $t = 0$ and $t = 1$, respectively. Where is it at $t = 2$?

The velocity is $v(t) = \int a(t)\,dt = \langle \frac{t^3}{3}, t^2, t - \frac{t^4}{4} \rangle + C$.

The position is $p(t) = \int v(t)\,dt = \langle \frac{t^4}{12}, \frac{t^3}{3}, \frac{t^2}{2} - \frac{t^5}{20} \rangle + Ct + D$.

Plugging in $t = 0$, we get $(0, 0, 0) = p(0) = \langle 0, 0, 0 \rangle + C(0) + D$, so $D = \langle 0, 0, 0 \rangle$.

Plugging in $t = 1$, we get $(1, 2, 3) = p(1) = (\frac{1}{12}, \frac{1}{3}, \frac{9}{20}) + C$, so $C = \langle \frac{11}{12}, \frac{5}{3}, \frac{51}{20} \rangle$.

Plugging in $t = 2$, we get $p(2) = (\frac{2^4}{12}, \frac{2^3}{3}, \frac{2^2}{2} - \frac{2^5}{20}) + 2C$

$= (\frac{16}{12}, \frac{8}{3}, \frac{4}{2} - \frac{32}{20}) + \langle \frac{11}{6}, \frac{10}{3}, \frac{51}{10} \rangle$

$= \boxed{(\frac{19}{6}, 6, \frac{11}{2})}.$