Introduction to (double) integrals

Written Project 3, due Friday, November 10

Just as the single integral $\int_{x=a}^{x=b} f(x)dx$ computes the area of the region above the interval $[a,b]$ and below the graph of $y = f(x)$, the double integral $\iint_R f(x,y)dydx$ computes the volume of the space above the region R and below the graph of $z = f(x,y)$.

We can think of the single integral as a “Riemann sum” of many rectangles, each having length dx and height $f(x)$; in the same way, the double integral can be thought of as a Riemann sum of many rectangular prisms, with length dx, width dy, and height $f(x,y)$.

When R is a rectangle with vertical and horizontal sides, the theory of the integral $\iint_R f(x,y)dxdy$ is thankfully very intuitive. Everything works out the way you'd hope it would, based on your knowledge of integration from Calculus I and II. When R is not a rectangle, things get messier — the theory doesn't change, but in practice it can get very difficult to even set up the problems. So let's stick to rectangles for now.

Directions

You may work in groups on this project, and you may turn in jointly written solutions in **groups of up to five**.

The expectations for your writeup are as follows:

1. All authors should be obviously listed as authors, and all sources (except for me or the textbook) should be clearly acknowledged. (See “Rules for written assignments” from the course syllabus or ask me if you have questions about this.)

   (In particular, you should **not expect any credit** for work that's been stuffed into the margins of this handout.)

2. Your work should be written in complete (mathematical) sentences, as if you were explaining the work to a confused classmate.

   Mathematical sentences, like English sentences, have a subject, a verb, and an object. In many cases, the verb is “=” (pronounced “equals” or “is equal to”, and only used when the subject and object are in fact equal).

   Even so, if you don't use any English words, you shouldn't expect full credit. (I know that many of you have never encountered an expectation like this in a math class before. If you aren't sure what I'm looking for, please plan to discuss it with me before the assignment is due.)

1. A scientist computes several values of a function $f(x)$:

x	3	4	5	6	9	11	13
f(x)	1	3	5	6	11	13	14

Use this data in a Riemann sum with at least five terms to approximate the value of $\int_{x=2}^{x=12} f(x)dx$.

(a) Decompose the interval $[2, 12]$ into at least five smaller intervals, each containing data. (There is no one "right" way to do this. The only rules are that the intervals must combine to cover the big interval $[2, 12]$ with no overlap; that there are at least five intervals; and that we need a value of $f(x)$ for some x in each interval (so that we can build the necessary rectangles.

The natural approach is to divide into five equal intervals, but there are other options.

(b) For each interval in the decomposition, do the following:

- Compute dx.
- Identify an x in the interval for which the value of $f(x)$ is known.
- Compute the area of the rectangle whose base is the interval, and whose height is $f(x)$. (**Do not average the values of $f(x)$ for multiple x.** Choose one x for each interval, and stick with it.

(c) Sketch a graph of the various rectangles above $[2, 12]$.

(d) Add the areas of the rectangles, and explain briefly (one sentence, maybe two) why this is a reasonable approximation of $\int_{x=2}^{x=12} f(x)dx$.

(a) Let's break up $[2, 12]$ into
$[2,4] \cup [4,6] \cup [6,8] \cup [8,10] \cup [10,12]$.

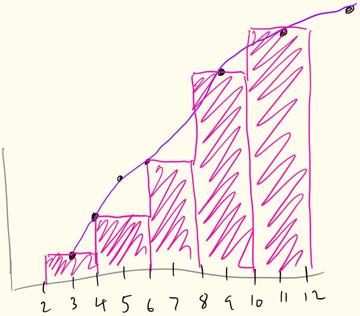
x	3	4	5	6	9	11	13
f(x)	1	3	5	6	11	13	14

(b)

Interval	dx	Choice of x	f(x)	Area = f(x)dx
[2,4]	2	3*	1	$1 \cdot 2 = 2$
[4,6]	2	4**	3	$3 \cdot 2 = 6$
[6,8]	2	6	6	$6 \cdot 2 = 12$
[8,10]	2	9	11	$11 \cdot 2 = 22$
[10,12]	2	11	13	$13 \cdot 2 = 26$

* 4 is also in [2,4], so $x = 4$, $f(x) = 3$ would also be ok.
* 5, 6 are also in [4,6] so $x=5$, $f(x) = 5$ or $x = 6$, $f(x) = 6$ would be okay.

(c)



(d) Our estimate is $\displaystyle\int_{x=2}^{x=12} f(x)\,dx \approx 2 + 6 + 12 + 22 + 26 = 68.$

This is reasonable, since the shaded rectangles are a decent approximation to the actual area under the curve.

If I'm trying to make the estimate as good as possible, I'll use more than five intervals, chosen to take advantage of all the data. But I'm going to start with three questions for the scientist:

(0) Theory says the graph of f is reasonably smooth, right? (e.g. if it's something like , the whole exercise makes no sense.)

(1) Is it okay to interpolate between $f(11)=13$ and $f(13)=14$ to get $f(12)\approx 13.5$?

(This isn't obvious — If it were $f(50)$ instead of $f(13)$, the answer would be no. And in any event there could be good reason to stop at 12. But let's assume this one is a "yes".)

(2) Is it okay to extrapolate from $f(4)=3$ and $f(3)=1$ to get $f(2)\approx -1$?

(This time, let's assume the negative value is nonsense, but we get permission to assume $f(2)\approx 0$.)

Then I'll put my data at the endpoints of my intervals, and use the trapezoid rule.

Interval	dx	f_{left}	f_{right}	$\frac{f_{left}+f_{right}}{2}$	Area $= \frac{f_{left}+f_{right}}{2}\,dx$
[2,3]	1	0	1	0.5	0.5
[3,4]	1	1	3	2	2
[4,5]	1	3	5	4	4
[5,6]	1	5	6	5.5	5.5
[6,9]	3	6	11	8.5	25.5
[9,11]	2	11	13	12	24
[11,12]	1	13	13.5	13.25	13.25

So the integral is about
$0.5+2+4+5.5+25.5+24+13.25$
$=74.75.$

2. Let $\mathcal{R}$ be the rectangle consisting of the points (x, y) with $0 \leq x \leq 12$, $10 \leq y \leq 30$. A scientist computes some values of the function $f(x, y)$ for points (x, y) inside $\mathcal{R}$:

y \ x	0	2	4	6	8	10	12
10	2	8	10	?	19	?	?
15	?	?	16	20	?	29	?
20	15	?	?	?	26	30	33
25	?	?	?	25	?	?	?
30	?	?	?	?	?	?	?

Use this data in a Riemann sum with at least five terms to approximate the value of $\iint_{\mathcal{R}} f(x)dx$.

(a) Decompose $\mathcal{R}$ into at least five smaller rectangles, each containing data. (As before, there is no one "right way" to do this. The requirements are that the rectangles comver $\mathcal{R}$ without overlapping, that there are at least five of them, and that each rectangle contains at least one data point. There's no need for all the rectangles to have the same shape, or even the same area.)

[Hint: **Draw your own picture!** When students attempt to superimpose a picture on the table, they almost universally make serious mistakes.]

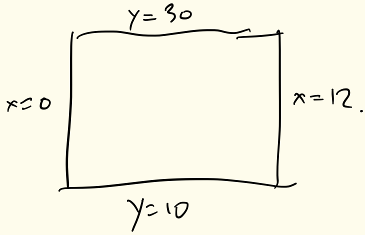
(b) Inside each rectangle in your decomposition, choose a sample point (x, y) for which the value of $f(x, y)$ is known.

(c) For each rectangle in your decomposition, compute the following:

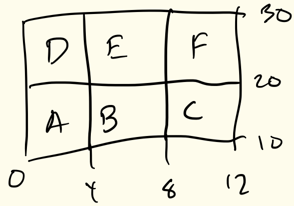
- The length dx.
- The width dy.
- The volume of the rectangular prism having the rectangle as its base, and $f(x, y)$ as its height.

(d) Add the volumes, and explain briefly (perhaps one sentence) why this is a reasonable approximation to $\iint_{\mathcal{R}} f(x)dx$.

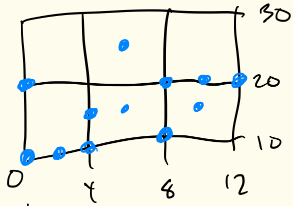
(a) The region R is



Let's chop it into six regions:



We have data at several points (marked with blue dots)



We need to choose one point from each region.

(b)(c)

Region	dx	dy	Chosen point P	f(P)	Volume = f(P) dy dx
A	4	10	(0, 10)	2	$2 \cdot 40 = 80$
B	4	10	(6, 15)	20	$20 \cdot 40 = 800$
C	4	10	(10, 15)	29	$29 \cdot 40 = 1160$
D	4	10	(0, 20)	15	$15 \cdot 40 = 600$
E	4	10	(6, 25)	25	$25 \cdot 40 = 1000$
F	4	10	(10, 20)	30	$30 \cdot 40 = 1200$

(d) So the integral is approximately $80 + 800 + 1160 + 600 + 1000 + 1200 = 4840$.

This is a reasonable approximation, because the rectangular prisms are a decent approximation for the area below the graph of $z = f(x,y)$.

3. Compute $\int_{x=0}^{x=3}\left[\int_{y=1}^{y=5} xy^2dy\right]dx.$ $= \int_{x=0}^{x=3}\left[\frac{xy^3}{3}\right]_{y=1}^{y=5} dx$

$$= \int_{x=0}^{x=3}\left(\frac{125x}{3} - \frac{x}{3}\right)dx$$

$$= \int_{x=0}^{x=3} \frac{124x}{3}\,dx$$

$$= \left[\frac{124x^2}{6}\right]_{x=0}^{x=3} = \frac{124}{6}(9-0) = 186$$

4. Compute $\int_{y=1}^{y=5}\left[\int_{x=0}^{x=3} xy^2dx\right]dy.$ $= \int_{y=1}^{y=5}\left[\frac{x^2y^2}{2}\right]_{x=0}^{x=3} dy$

$$= \int_{y=1}^{y=5}\left(\frac{9y^2}{2} - \frac{0y^2}{2}\right)dy$$

$$= \int_{y=1}^{y=5} \frac{9y^2}{2}\,dy$$

$$= \left[\frac{9y^3}{6}\right]_{y=1}^{y=5} = \frac{375}{2} - \frac{3}{2} = 186.$$

5. Did you get the same numerical answer to problems 3 and 4? Approximate each of the double integrals with a (Riemann sum of) Riemann sum(s), and compare the two sums. Use these sums to explain why the two integrals should (or shouldn't) be equal to each other.

(You might find it worthwhile to chop the region into 12 squares, and find a way to approximate both integrals with prisms above these squares.)

They are the same.

We can approximate the first integral by breaking the interval $[0,3]$ up into $[0,1]\cup[1,2]\cup[2,3]$; then (with sample x-values of 0.5, 1.5, 2.5 and $dx=1$)

$$\int_{y=1}^{y=5}\frac{1}{2}y^2\,dy + \int_{y=1}^{y=5}\frac{3}{2}y^2\,dy + \int_{y=1}^{y=5}\frac{5}{2}y^2\,dy$$

$$\approx \begin{pmatrix} \frac{1}{2}\left(\frac{3}{2}\right)^2 \\ +\frac{1}{2}\left(\frac{5}{2}\right)^2 \\ +\frac{1}{2}\left(\frac{7}{2}\right)^2 \\ +\frac{1}{2}\left(\frac{9}{2}\right)^2 \end{pmatrix} + \begin{pmatrix} \frac{3}{2}\left(\frac{3}{2}\right)^2 \\ +\frac{3}{2}\left(\frac{5}{2}\right)^2 \\ +\frac{3}{2}\left(\frac{7}{2}\right)^2 \\ +\frac{3}{2}\left(\frac{9}{2}\right)^2 \end{pmatrix} + \begin{pmatrix} \frac{5}{2}\left(\frac{3}{2}\right)^2 \\ +\frac{5}{2}\left(\frac{5}{2}\right)^2 \\ +\frac{5}{2}\left(\frac{7}{2}\right)^2 \\ +\frac{5}{2}\left(\frac{9}{2}\right)^2 \end{pmatrix}.$$

Similarly, the second integral is approximated by

$$\begin{bmatrix} \int_{x=0}^{x=3} x\left(\frac{3}{2}\right)^2dx \\ +\int_{x=0}^{x=3} x\left(\frac{5}{2}\right)^2dx \\ +\int_{x=0}^{x=3} x\left(\frac{7}{2}\right)^2dx \\ +\int_{x=0}^{x=3} x\left(\frac{9}{2}\right)^2dy \end{bmatrix} \approx \begin{array}{l} \left[\frac{1}{2}\left(\frac{3}{2}\right)^2 + \frac{3}{2}\left(\frac{3}{2}\right)^2 + \frac{5}{2}\left(\frac{3}{2}\right)^2\right] \\ +\left[\frac{1}{2}\left(\frac{5}{2}\right)^2 + \frac{3}{2}\left(\frac{5}{2}\right)^2 + \frac{5}{2}\left(\frac{5}{2}\right)^2\right] \\ +\left[\frac{1}{2}\left(\frac{7}{2}\right)^2 + \frac{3}{2}\left(\frac{7}{2}\right)^2 + \frac{5}{2}\left(\frac{7}{2}\right)^2\right] \\ +\left[\frac{1}{2}\left(\frac{9}{2}\right)^2 + \frac{3}{2}\left(\frac{9}{2}\right)^2 + \frac{5}{2}\left(\frac{9}{2}\right)^2\right]. \end{array}$$

The sums are the same, so the integrals are the same.