The multivariate chain rule

Written Project 2, due Friday, November 3

The **multivariate chain rule** says that, if z is a function of x and y, then

$$dz = z_x dx + z_y dy,$$

whenever P and $P + \langle dx, dy, dz \rangle$ are nearby points on the graph of z. (This is equivalent to saying that $\langle dx, dy, dz \rangle$ is a tangent vector to the graph of z at P.)

The chain rule also says that, if F is a function of x, y, and z, then

$$dF = F_x dx + F_y dy + F_z dz$$

whenever P and $P + \langle dx, dy, dz \rangle$ are nearby points.

The chain rule interacts usefully with the standard rules of algebra, so for example if $f = g$ on some curve (in $\mathbb{R}^2$) or surface (in $\mathbb{R}^3$), then $df = dg$ for any two nearby points on that curve or surface. If we have formulas for f and g, then substitution yields a relationship among dx, dy, etc. This can be particularly useful if f or g is a constant (e.g., $f(x, y) = 2$, because $d(2) = 0$).

Essentially every problem involving (first) derivatives can be solved by a very direct application of the chain rule. On this assignment, I am asking you to solve several problems in this way.

Directions

You may work in groups on this project, and you may turn in jointly written solutions in **groups of up to five**.

The expectations for your writeup are as follows:

1. All authors should be obviously listed as authors, and all sources (except for me or the textbook) should be clearly acknowledged. (See "Rules for written assignments" from the course syllabus or ask me if you have questions about this.)

2. Your work should be written in complete (mathematical) sentences, as if you were explaining the work to a confused classmate.

   Mathematical sentences, like English sentences, have a subject, a verb, and an object. In many cases, the verb is "=" (pronounced "equals" or "is equal to", and only used when the subject and object are in fact equal).

   Even so, if you don't use any English words, you shouldn't expect full credit. (I know that many of you have never encountered an expectation like this in a math class before. If you aren't sure what I'm looking for, please plan to discuss it with me before the assignment is due.)

3. The assignment is due at Friday's exam. That said, if you turn it in on Monday, I will grade it and return it on Wednesday. And if you bring it to office hours at any time during the week (and office hours aren't crowded), I'll be happy to grade it on the spot.

1. Find the equation of the tangent plane to the graph of $z = 4x^2 - y^2 + 2y$ at $P = (-1, 2, 4)$.

   (a) Write out the chain rule for z as a function of x and y.

   $dz = z_x\,dx + z_y\,dy$

   (b) What are the values of z_x and z_y at P?

   $z_x = 8x$

   and $z_y = -2y + 2$.

   Plugging in $x = -1$, $y = 2$ from P, we get

   $z_x(P) = 8(-1) = -8$ and $z_y(P) = -2(2) + 2$
   $= -2$

   (c) What are the values of dx, dy, and dz are a point (x, y, z) which is near P and on the graph of z?

   $\langle dx, dy, dz \rangle = (x, y, z) - (-1, 2, 4)$
   $= \langle x+1,\ y-2,\ z-4 \rangle$.

   So $dx = x+1$, $dy = y-2$, and $dz = z-4$.

   (d) Plug your answers to parts (b) and (c) in to the chain rule from part (a).

   $dz = z_x\,dx + z_y\,dy$, so

   $\boxed{(z-4) = (-8)(x+1) + -2(y-2)}$.

2. Find the equation of the tangent line to the level curve of $f(x, y) = e^{-xy}\cos(x + y)$ that passes through $P = (\pi, 0)$.

(a) Write out the chain rule for f as a function of x and y.

$df = f_x dx + f_y dy.$

(b) What are the values of f_x and f_y at P?

$f_x = -ye^{xy}\cos(x+y) + e^{-xy}(-\sin(x+y))$

and $f_y = -xe^{xy}\cos(x+y) + e^{-xy}(-\sin(x,y)).$

Plugging in $x=\pi, y=0$, we get $f_x(P) = 0 + e^0(-\sin\pi) = 0$

and $f_y(P) = -\pi e^0\cos(\pi) + e^0(-\sin\pi) = \pi.$

(c) What are the values of df, dx, and dy are a point (x, y) which is near P and on the level curve? (You will probably need some English to explain the value of df.)

$df = f(x,y) - f(P)$. Since (x,y) and P are on the same level curve, we know $f(x,y) = P$.

Thus $\boxed{df = 0}$.

Meanwhile, $\langle dx, dy\rangle = (x,y) - P$,

so $\boxed{dx = x - \pi}$ and $\boxed{dy = y - 0}$

(d) Plug your answers to parts (b) and (c) in to the chain rule from part (a).

We have $df = f_x dx + f_y dy$

i.e. $\boxed{0 = 0(x-\pi) + \pi(y-0)}$

i.e. $0 = \pi y$

i.e. $y = 0.$

3. Find the equation of the tangent line to the level surface of $F(x, y, z) = 2x^2 + 3y^2 - 5z^2$ that passes through $P = (1, 0, -6)$.

(a) You have a choice here: You could solve for z and treat this like problem 1, or you could leave it as a level surface and treat it like problem 2.

(b) Depending on your choice in (a), either write out the chain rule for F as a function of x, y, and z, or solve for z as a function of x and y, and then write out the associated chain rule.

The easier solution is to leave it as written.
Then

$dF = F_x\,dx + F_y\,dy + F_z\,dz$

At P, $F = 2 - 180 = -178$.
So the level curve has equation

$2x^2 + 3y^2 - 5z^2 = -178$

$z^2 = \dfrac{2x^2+3y^2+178}{5}$

$z = -\sqrt{\dfrac{2x^2+3y^2+178}{5}}$, and $dz = z_x\,dx + z_y\,dy$.
(minus because $z = -6$).

(c) Compute the values (at P) of every term appearing in the chain rule, and use them to finish the problem.

$dF = 0$

$F_x = 4x$, so $F_x(P) = 4$

$dx = x - 1$

$F_y = 6y$, so $F_y(P) = 0$

$dy = y - 0$

$F_z = -10z$, so $F_z(P) = 60$

$dz = z + 6$.

We get

$0 = 4(x-1) + 0(y-0) + 60(z+6)$.

We get

$z_x = \dfrac{-\frac{4x}{5}}{2\sqrt{\frac{2x^2+3y^2+178}{5}}}$ and $z_y = \dfrac{-\frac{6y}{5}}{2\sqrt{\frac{2x^2+3y^2+178}{5}}}$

At P, this is

$z_x = -\dfrac{4}{60} = -\dfrac{1}{15}$. and $z_y = 0$

And $\langle dx, dy, dz\rangle = \langle x-1, y-0, z+6\rangle$
So the plane is

$$\boxed{z + 6 = -\frac{1}{15}(x-1) + 0(y-0)}$$

4. Use the chain rule to compute $\frac{dz}{dt}$, where

$$z = \left(e^t \sin t\right)^2 \left(t^2 + t\right) + 3\left(e^t \sin t\right)\left(t^2 + t\right)^4.$$

(You must use the multivariate chain rule here to earn any credit. I realize that you could just multiply everything out and use Calculus I techniques — but I could have prevented that by making this messier. Humor me.)

(a) Introduce useful quantities x and y, and rewrite z as a function of x and y. If we set $x = e^t \sin t$ and $y = t^2 + t$,

then $z = x^2 y + 3xy^4$.

(b) Write out the chain rule for z as a function of x and y.

$dz = z_x dx + z_y dy$

(c) Do algebra to the chain rule so that it has a $\frac{dz}{dt}$ instead of just a dz.

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$$

(d) Compute z_x, z_y, $\frac{dx}{dt}$, and $\frac{dy}{dt}$.

$z_x = 2xy + 3y^4$, $\quad z_y = x^2 + 12xy^3$,

$\frac{dx}{dt} = e^t \sin t - e^t \cos t$, $\quad \frac{dy}{dt} = 2t + 1$

(e) Use your answers to part (d) to compute $\frac{dz}{dt}$ in terms of x, y, and t.

$$\frac{dz}{dt} = \left(2xy + 3y^4\right)\left(e^t \sin t - e^t \cos t\right) + \left(x^2 + 12xy^3\right)(2t+1).$$

(f) Rephrase your solution in a boss-friendly manner. (Remember: your boss posed the problem with z as a function of only t, and doesn't have the time and/or skill to read your work.)

There are two reasonable options. One is to provide the context:

$$\boxed{\frac{dz}{dt} = \left(2xy + 3y^4\right)\left(e^t \sin t - e^t \cos t\right) + \left(x^2 + 12xy^3\right)(2t+1), \text{ where } x = e^t \sin t \text{ and } y = t^2 + t}$$

The other is to substitute:

$$\boxed{\frac{dz}{dt} = \left[2\left(e^t \sin t\right)\left(t^2 + t\right) + 3\left(t^2 + t\right)^4\right]\left(e^t \sin t - e^t \cos t\right) + \left[\left(e^t \sin t\right)^2 + 12\left(e^t \sin t\right)\left(t^2 + t\right)^3\right](2t+1).}$$

5. Let $z = 1 + x\sqrt{y}$ and $\mathbf{v} = \langle 4, -3 \rangle$. Find the directional derivative $D_{\mathbf{v}}(z)$.

(a) Write out the chain rule for z as a function of x and y.

$$dz = z_x\,dx + z_y\,dy.$$

(b) Do algebra to the chain rule so that it has a $\frac{dz}{dt}$ instead of just a dz.

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}.$$

(c) Determine $\frac{dx}{dt}$ and $\frac{dy}{dt}$ when moving in the direction of $\mathbf{v}$. Explain. (Remember, it's your choice whether to interpret $\mathbf{v}$ as a direction vector (so moving with unit speed) or as a velocity vector. Our textbook says to interpret it as a direction vector, but some other texts disagree.)

If v is a velocity vector, then $v = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle$, so $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = -3$

If v is a direction vector, then $\langle \frac{dx}{dt}, \frac{dy}{dt} \rangle = \frac{v}{\|v\|}$, so $\frac{dx}{dt} = \frac{4}{5}$ and $\frac{dy}{dt} = -\frac{3}{5}$.

(d) Compute z_x and z_y.

$$z_x = \sqrt{y} \text{ and } z_y = \frac{x}{2\sqrt{y}}.$$

(e) Make the necessary substitions to determine $D_{\mathbf{v}}(z) = \frac{dz}{dt}$.

$$\frac{dz}{dt} = 4(\sqrt{y}) - 3\left(\frac{x}{2\sqrt{y}}\right) = 4\sqrt{y} - \frac{3x}{2\sqrt{y}}$$

$$\frac{dz}{dt} = \frac{4}{5}(\sqrt{y}) - \frac{3}{5}\left(\frac{x}{2\sqrt{y}}\right) = \frac{4\sqrt{y}}{5} - \frac{3x}{10\sqrt{y}}.$$