Linear Geometry in 3-Space

Written Project 1, due Wednesday, September 13

There are many ways to use two points, lines, and/or planes to define a third such object. We want to be able to find the third object in every such situation.

The correct learning process here is not to memorize every procedure, as there are simply too many. Instead, we want to get good at thinking about the geometry of the situation, and understand the tools at our disposal. I have two generic recommendations:

- The cross product of two vectors is perpendicular to both. This makes it a very powerful tool. Recognize that power, because you'll want to use it over and over again.

- Draw or build a model of your situation, ignoring all equations and coordinates. (By "build a model" I mean to actually construct the situation out of appropriate materials, like sticks and construction paper, or play-doh or something.) This will help you visualize the relationship between things that you know and things that you don't know yet.

There are several ways to find almost all the points, lines, and planes we're interested in; in every situation I have outlined one. You need not slavishly follow the outline; in fact there is frequently a slightly more efficient method available and I encourage you to look for it.

Directions

There's too much here to ask you to write it all carefully, but there's not too much to ask you to think through the plans for every question. The handout is divided into seven sections, and I want you to turn in two.

At the start of class on Friday, September 8, I will randomly choose one section for myself to post model solutions, and one for everyone to turn in. You may choose your second section from among the other five (but be very clear about which you have chosen).

You may work in groups on this project, and you may turn in jointly written solutions in groups of up to five.

The expectations for your writeup are as follows:

1. All authors should be obviously listed as authors, and all sources (except for me or the textbook) should be clearly acknowledged. (See "Rules for written assignments" from the course syllabus or ask me if you have questions about this.)

2. Each of the sections you're turning in should get a separate, clearly labeled page, unless one of them requires multiple pages. In that case, there should be a page break between the sections.

   (In particular, you should **not expect any credit** for work that's been stuffed into the margins of this handout.)

3. Your work should be written in complete (mathematical) sentences, as if you were explaining the work to a confused classmate. In general, if you don't use any English words, you shouldn't expect much credit. (I know that many of you have never encountered an expectation like this in a math class before. If you aren't sure what I'm looking for, please plan to discuss it with me before the assignment is due, and/or see the model solutions that will be posted to the course web page between September 8 and September 10.)

4. You should solve all the numbered problems in you section (although you're welcome to solve them out of order if that makes the explanations easier.) Somewhere along the way you should probably be restating those questions — otherwise the write-up won't make sense.

   That said, you are welcome to ignore the lettered sub-questions. They are there as suggestions, but you are reasonably likely to find other paths that you view as easier to understand, more efficient, or both.

1 Two points

Given two points P and Q, there is a line ℓ passing through both points. The set of points which are equidistant from P and Q forms a plane L, which includes R, the midpoint of $\overline{PQ}$. We'd like to find all three.

Let $P = (-1, 8, -5)$ and $Q = (13, 18, -11)$.

1. Find ℓ.
   (a) Find a vector parallel to ℓ.
   (b) Identify a point on ℓ.
   (c) Find a parametrization for ℓ.
2. Find R.
   (a) In your parametrization for ℓ, what is t at P? at Q?
   (b) What is t at R?
   (c) Find R.
3. Find L.
   (a) Convince yourself that L is really a plane, and that it really contains R.
   (b) Convince yourself that ℓ and L are perpendicular.
   (c) Find a normal vector to L.
   (d) Find numbers A, B, and C such that the equation for L has the form $Ax + By + Cz = D$.
   (e) Plug in R to find D.

2 A point and a line

Given a line ℓ and a point P *not* on ℓ, there is a unique plane L containing ℓ and P. Euclid's fifth postulate says there is another line m in L parallel to ℓ and passing through P. We'd also like to know the distance from P to ℓ, the coordinates of the point Q on ℓ which is as close to P as possible, and the equation of the line n passing through P and Q.

Let $P = (-3, -3, 3)$ and let ℓ be defined by the parametrization $(x, y, z) = (2, 4, -3) + t(0, -3, 1)$.

1. Verify that P is not on ℓ.

2. Find m.

   (a) Identify a point on m. (We only know one such point at this time. If you're struggling, you're trying to make it too hard.)

   (b) Identify a vector in the direction of m. (Again, if you're struggling, you're trying to make it too hard.)

   (c) Find a parametrization for m.

3. Find L.

   (a) Find two vectors parallel to L. (If necessary, start by identifying three points on L.)

   (b) Find a normal vector for L.

   (c) Identify numbers A, B, and C such that the equation for L has the form $Ax + By + Cz = D$.

   (d) Plug P into the above equation to determine D.

   (e) Write down an equation for L.

4. Find n.

   (a) Draw a picture with P, Q, and ℓ in it.

   (b) Identify a vector in the picture which is perpendicular to $n = PQ$.

   (c) Identify another vector perpendicular to n. (Think outside of the plane.)

   (d) Find a vector parallel to n. (Use your answers from (b) and (c).)

   (e) Find a parametrization for n. (You just found a direction vector. You also know a point.)

5. Find Q.

   (a) Find the intersection between ℓ and n.

6. Find the distance from P to ℓ.

   (a) Use the Pythagorean theorem to find the distance from P to Q.

(b) Find an equation $Ax + By + Cz = D$ for the plane through P with normal vector parallel to n.

(c) Find an equation $Ax + By + Cz = D'$ for the plane through ℓ with normal vector parallel to n.

(d) What does the distance from P to Q have to do with A, B, C, D, and D'? Explain how this observation could lead to an easier method for finding distance.

3 A point and a plane

Given a plane L and a point P *not* on L, there is another plane M parallel to L but containing P. We'd also like to know the distance from P to L, the point Q on L which is as close as possible to P, and the line ℓ passing through P and Q.

Let $P = (4, 3, -3)$ and let L be defined by the equation $x + 5y - 2z = -3$.

1. Verify that P does not lie on L.
2. Find M.
   (a) Identify a vector normal to L.
   (b) Identify a vector normal to M.
   (c) Identify a point on M.
   (d) Find an equation for M.
3. Find ℓ.
   (a) Explain why ℓ is perpendicular to L.
   (b) Identify a vector in the direction of ℓ.
   (c) Find a parametrization for ℓ.
4. Find Q.
   (a) Find the intersection of L with ℓ.
5. Find the distance from P to L.
   (a) Use the Pythagorean theorem to find the distance from P to Q.
   (b) Can you extract this distance somehow from the equations of L and M, without first knowing Q? (It may help to look at the magnitude of the normal vector.)

4 Two lines in the same plane

Two lines ℓ and m that share a plane are either parallel or intersecting. When this happens, we'd like to know the equation of the shared plane. If they're parallel, we'd also like to know the distance between them, and the equation of another line in the plane that's perpendicular to both.

4.1 Parallel lines

Let ℓ be defined by the parametrization $(x, y, z) = (-2, -1, 3) + t(-3, -5, -5)$ and let m be defined by the parametrization $(x, y, z) = (2, 1, 2) + t(3, 5, 5)$.

1. Verify that the lines are parallel and do not intersect.
2. Find the plane L containing ℓ and m.
   (a) Identify two vectors in L. (One vector should be obvious. If you're having trouble finding another, remember that you know one point on each of the given lines.)
   (b) Find a vector normal to L.
   (c) Identify a point on L. (You should have no trouble finding two. Pick one of those.)
   (d) Find an equation for L.
3. Find a line n inside L that's perpendicular to ℓ and m.
   (a) Choose a point P in L.
   (b) Build the situation using physical objects, and draw in an appropriate line n through your point P.
   (c) Identify a vector inside L that's perpendicular to n.
   (d) Identify a vector outside L that's perpendicular to n.
   (e) Find a vector parallel to n.
   (f) Find a parametrization for n.
4. Find the distance from ℓ to m.
   (a) Compute the intersection Q of ℓ and n.
   (b) Compute the intersection R of m and n.
   (c) Use the Pythagorean theorem to find the distance from Q to R.

4.2 Intersecting lines

Let ℓ be defined by the parametrization $(x, y, z) = (-2, 4, 0) + t(-2, 3, -2)$ and m be defined by the parametrization $(x, y, z) = (-6, 7, -8) + t(3, -3, 5)$.

1. Verify that ℓ and m intersect, and find their point of intersection P.
2. Find the plane L containing ℓ and m.
   (a) Identify two vectors in L.
   (b) Find a vector normal to L.
   (c) Identify a point in L.
   (d) Find an equation for L.

5 Skew lines

Two lines that don't intersect and aren't parallel are called **skew**. Skew lines define two parallel planes, each containing one line and parallel to the other. They also define a unique line which passes perpendicularly through both. This intersects the two lines at a point P on ℓ and a point Q on m which are as close to each other as possible. We'd like to find all of these things, and the distance between the two lines.

Let ℓ be defined by the parametrization $(x, y, z) = (5, -1, -2) + t(5, -3, 4)$ and let m be defined by the parametrization $(x, y, z) = (0, 5, -1) + t(-1, 5, 1)$.

1. Verify that ℓ and m do not intersect and are not parallel.
2. Find parallel planes L and M containing ℓ and m, respectively.
   (a) Find a vector parallel to ℓ.
   (b) Find a vector parallel to m.
   (c) Explain why both of these vectors are parallel to L and M.
   (d) Find a vector normal to both L and M.
   (e) Find a point on L.
   (f) Find the equation for L.
   (g) Find a point on M.
   (h) Find the equation for M.
3. Find P, Q, and the line n passing through both.
   (a) There are many, many ways to do this one, and they all involve large numbers of steps, messy computations, or both. I encourage you in the strongest possible terms to build the situation out of physical objects and figure out everything you can before reading the remaining hints. A method that you figure out on your own is a method that you'll understand better.
   (b) Explain why n is perpendicular to both L and M.
   (c) Find a vector $\mathbf{v}$ parallel to n.
   (d) Find a point R on ℓ and a point S on m.
   (e) Compute $\text{proj}_{\mathbf{v}}(RS)$, and explain why this is equal to PQ.
   (f) Use the parametrization for ℓ to find an expression for P involving a parameter t.
   (g) Use the parametrization for m to find an expresssion for Q involving a parameter s.
   (h) Use the fact that $Q = P + PQ$ to find an expression for Q involving t.

(i) We now have two expressions for Q, which together constitute three equations in two variables. Solve them simultaneously.

(j) Find P and Q.

(k) Find a parametrization for n.

4. Find the distance from ℓ to m.

   (a) Use the Pythagorean theorem to compute the distance from P to Q.

   (b) Could you have found the this distance from the equations for L and M without first knowing P and Q?

6 A line and a plane

A line ℓ and a plane L are either parallel or intersecting. If they're parallel, there's a plane M containing ℓ which is parallel to L. We'd also like to know the distance from ℓ to L. If they intersect, we'd like to know the point and angle of intersection.

6.1 The intersecting case

Let ℓ be defined by the parametrization $(x, y, z) = (-1, 2, 2,) + t(5, 4, 5)$ and let L be defined by the equation $3x + 5y - z = 3$.

1. Verify that ℓ is not contained in L.

2. Find the point of intersection.

   (a) Identify four equations in the four variables x, y, z, and t.

   (b) Solve for x, y, and z. (It's probably easiest to first solve for t, but if you can find the other three variables without finding t that's fine.)

3. Find the angle of intersection.

   (a) Identify a vector normal to L.

   (b) Identify a vector parallel to ℓ.

   (c) Find the angle between these two vectors.

   (d) Explain the relationship between this angle and the desired angle between ℓ and L.

   (e) Find the angle between ℓ and L.

6.2 The parallel case

Let ℓ be defined by the parametrization $(x, y, z) = (-2, -3, 2) + t(-1, 5, 2)$ and let L be defined by the equation $x + 3y - 7z = -2$.

1. Verify that ℓ and L are parallel.

2. Find an equation for M.

   (a) Identify a vector normal to L.

   (b) Find a vector normal to M.

   (c) Identify a point on M.

   (d) Find an equation for M.

3. Find the distance from ℓ to L.

   (a) Find a point P on M.

   (b) Find a parametrization of the line m through P perpendicular to L.

(c) Find the intersection Q of L and m.

(d) Use the Pythagorean theorem to find the distance from P to Q.

(e) Could you have found this distance just from the equations of L and M, without doing all that work?

7 Two planes

Two planes L and M either intersect or are parallel. If they're parallel, we'd like to know the distance between them. If they intersect, we'd like to find the line and angle of intersection, and the equation of the line of intersection.

7.1 The parallel case

Let L be defined by the equation $2x + 3y + 2z = 2$ and let M be defined by the equation $-6x - 9y - 6z = 3$.

1. Find the distance from L to M.

   (a) Find a point P on M.

   (b) Find a parametrization of the line ℓ through P that is perpendicular to L.

   (c) Find the point Q where ℓ passes through L.

   (d) Use the Pythagorean theorem to find the distance from P to Q.

2. Explain how to find the distance from L to M without computing any other planes or lines.

   (a) Skim the other parts of the assignment, and observe that many of your classmates would really like a solution to this.

   (b) Scale the equations for L and/or M so that they both have the same coefficients on x, y, and z.

   (c) L and M now have the same "natural" choice of normal vector. Do you see how it shows up in your answer to the previous problem?

   (d) How do the constant terms show up?

   (e) Make a guess about how to easily read off the distance between two parallel planes from just their equations. Then make up a bunch of pairs, and test your guess by computing the distance carefully as in problem 1.

7.2 The intersecting case

Let L be defined by the equation $3x - y = -1$ and let M be defined by the equation $2x - y + 4z = 3$.

1. Verify that L and M are different.

   (a) Either find a point that's on one plane but not the other, or explain how you know the normal vectors are different.

2. Find the angle of intersection.

   (a) Identify normal vectors to L and to M.

(b) Find the angle between these normal vectors.

(c) Explain why the angle between the normal vectors is the same as the angle between the planes. (Build two planes out of paper or books or something, and rotate the assembly around the line of intersection.)

3. Find the line of intersection ℓ.

   (a) Explain why the normal vectors to L and M are both perpendicular to ℓ.

   (b) Find a vector parallel to ℓ.

   (c) Find a point on ℓ. (This point satisfies the equations for L and M, which is two equations in three variables. How can we find a third equation?)

   (d) Find a parametrization for ℓ.