Math 2163

Jeff Mermin's section, Extra credit Written project 4, due Monday, December 11 (at the final exam)

Preamble:

In semesters where we don't study Lagrange multipliers, I use this written project to supplement chapters 15.4 and 15.6 on changes of variable in double and triple integrals. You may (and **should**) work in groups of up to five, and turn in jointly written solutions. It will be worth roughly half as much as a regular written assignment.

The handout is in two parts. The first part (sections §1 through §5) is some exposition on so-called "exterior differential algebras", with some exercises mixed in. These exercises are purely for practice: You should make sure you can do them so you're comfortable with the material, but you shouldn't plan to write them up carefully or turn them in. I will post solutions (in the sense of odd-numbered problems at the back of the book) over the first weekend.

The second part (on the last page) consists of problems to solve and write up carefully. You should write up each problem in a way that makes it as easy to read as possible (probably using a separate page for each problem).

I recommend that you approach the material as follows: First, read Chapter 15.4 in the text and become comfortable with polar and spherical coordinates. Second, read sections §1 through §3 of this handout, and work at least some of the exercises contained in it. Finally, read Chapter 15.6 in conjunction with §4 and §5, carefully working the examples. (Once you've done all of that, solve the problems on page 9, and turn in your solutions.)

1 Motivating Examples

Let's start with a couple of motivating examples, involving double integrals and the area of parallelograms.

Example 1. Let W be the square with vertices $(\pm 1, 0)$ and $(0, \pm 1)$. Find its area using high-school geometry, then using integrals.

(a) Verify that W is in fact a square, and compute the lengths and equations of its sides. Use these to determine its area.

(b) Explain why the area of the square is $\iint\limits_W 1dydx$. Then express that area using iterated integrals in the variables x and y, with the x on the outside integral.

This should be annoying because the upper and lower boundaries change on you halfway through. You can dodge the annoyance using symmetry here, but in general that's a bad strategy (what if we were taking the integral of some complicated and non-symmetric function like $(x+y)dydx$ instead of just $1dydx$?). Still, compute the integrals and verify that you get the right answer.

(c) Compute the area of the square using iterated integrals in x and y, with the y on the outside integral. This should be annoying in the same way as before. Still, compute the integrals and verify that you're getting the right answer.

(d) Let's try to be clever in a different way. Let $u = x + y$ and $v = x - y$. Verify that the sides of the square have equations $u = \pm 1$ and $v = \pm 1$. So maybe we can do a u, v-substitution?

The area is $\iint 1dydx$. In translating to u and v, that "$\iint$" becomes "$\int_{u=-1}^{u=1} \int_{v=-1}^{v=1}$", and the "1" becomes "1", but what are we to do with the "$dydx$"? After all, $\int_{u=-1}^{u=1} \int_{v=-1}^{v=1} 1dydx$ doesn't make sense because the boundaries don't match the variables. We might naively hope that we can just swith $dydx$ to $dvdu$.

Evaluate $\int_{u=-1}^{u=1} \int_{v=-1}^{v=1} 1dvdu$, and verify that we get the wrong answer.

(e) One way to have seen in advance that the computation above was wrong is to notice (using the techniques of chapter 12) that the lines $u = 1$ and $u = -1$ aren't 2 units apart; they're $\frac{1-(-1)}{\sqrt{1^2+1^2}}$ units apart. So we might guess that du and dv are each "too big" by a factor of $\sqrt{2}$. Indeed, dividing out by two copies of $\sqrt{2}$ gives us the right answer. So that looks like it might be a working fix. Unfortunately, it requires that du and dv are perpendicular, as the next example will show.

Example 2: Let W be the parallelogram with vertices $P = (1, -9)$, $Q = (9, -15)$, $R = (-1, 9)$, and $S = (-9, 15)$. Compute the area of W using classical geometry, then using integrals.

(a) Verify that W is a parallelogram, and find the equations for its sides. Compute the area of W using either cross products or techniques from high school geometry.

(b) Express the area of W using iterated integrals in x and y, with x on the outside integral. (This should be even more annoying than in the first example. You don't have any obvious symmetry to exploit, and the top and bottom boundaries break at different x-values, so you just can't get away from having three integrals.

(c) Express the area of W using iterated integrals in x and y, with y on the outside integral.

(d) Evaluate one of your two expressions above, and make sure you get the right answer.

(e) Look at the equations you found for the sides of W, and find suitable coordinates u and v so that "$\displaystyle\iint_W$" can be rewritten as a single iterated integral using u and v. Then rewrite $\displaystyle\iint_W 1dxdy$ in this form.

(f) Verify that, as in the previous example, just replacing $dxdy$ with $dudv$ gives us the wrong answer.

(g) Last time, we naively just divided by the lengths of two vectors associated with u and v. Compute the corresponding lengths this time, and verify that dividing out still gives the wrong answer.

(h) The wrong answer from (f) above is off by a factor. Can you see how to get the missing factor out of the two vectors in (g)?

The fix, which turns out to work in general, is that we want to essentially take a cross product of du and dv. (This will give us an additional multiplicative factor of the sine of the angle between the directions described by du and dv.) The problem is that du and dv aren't vectors (and certainly aren't vectors in $\mathbb{R}^3$) in any meaningful sense, so how can we take their cross product?

When mathematicians encounter a problem like this, our response is generally to invent an entirely new branch of mathematics in which the things can be made to behave like we want. In this case, the new field is called "exterior algebra", and it turns differentials into vector-like objects, with a multiplication that's sort of like the cross product. It turns out that the key property of the cross product is its anticommutativity: $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$. We can derive everything else from there.

2 Exterior differential algebras

An *exterior differential algebra* is an abstraction which simultaneously generalizes several multidimensional concepts: cross products, determinants, tangent vectors, and double and triple integrals, to name a few.

Elements of an exterior differential algebra are functions f, "differential symbols" $[\mathbf{dg}]$ for differentiable functions g, and anything that we can get by adding and multiplying these things. Two typical elements of the exterior differential algebra in the single variable x are

$$\begin{aligned} A &= \sin(x)[\mathbf{d(e^x)}] - x^3[\mathbf{d(x^2)}] \\ B &= 4x[\mathbf{d(x^5-1)}]. \end{aligned}$$

Their sum and product are

$$\begin{aligned} A + B &= \sin(x)[\mathbf{d(e^x)}] - x^3[\mathbf{d(x^2)}] + 4x[\mathbf{d(x^5-1)}] \\ AB &= 4x\sin(x)[\mathbf{d(e^x)}][\mathbf{d(x^5-1)}] - 4x^4[\mathbf{d(x^2)}][\mathbf{d(x^5-1)}]. \end{aligned}$$

Note that subtraction makes sense but division does not, since for example we can't multiply A by anything to get 1.

The differential symbols interact with functions via the chain rule

$$[\mathbf{df}] = \tfrac{df}{dx}[\mathbf{dx}].$$

Thus, for example, we have

$$\begin{aligned} A &= \sin(x)[\mathbf{d(e^x)}] - x^3[\mathbf{d(x^2)}] \\ &= \sin(x)(e^x[\mathbf{dx}]) - x^3(2x[\mathbf{dx}]) \\ &= (e^x\sin(x) - 2x^4)[\mathbf{dx}]. \end{aligned}$$

Example 3. Let $f(x) = e^x \sin x$ and $g(x) = x \ln x$. Simplify the following as much as possible:

(a) $[\mathbf{df}]$

(b) $[\mathbf{dg}]$

(c) $f[\mathbf{dg}] + g[\mathbf{df}]$

(d) $[\mathbf{d(fg)}]$ (Why is this not surprising?)

(e) $[\mathbf{df}][\mathbf{dg}]$

In more variables, we use the multivariate chain rule, so, for example,

$$[\mathbf{d}(\sqrt{\mathbf{x}+\mathbf{y^2}})] = \frac{1}{2\sqrt{x+y^2}}[\mathbf{dx}] + \frac{y}{\sqrt{x+y^2}}[\mathbf{dy}].$$

Example 4. Let $F(x,y) = x^2 + y^2 - xy$ and $G(x,y,z) = xye^z$. Simplify the following as much as possible.

(a) $[\mathbf{dF}]$

(b) $[\mathbf{dG}]$

(c) $F[\mathbf{dG}] + G[\mathbf{dF}]$

(d) $[\mathbf{d(FG)}]$

(e) $[\mathbf{dF}][\mathbf{dG}]$

The differential symbols are *anticommutative*; that is,

$$[\mathbf{df}][\mathbf{dg}] = -[\mathbf{dg}][\mathbf{df}].$$

These rules allow us to simplify any element of an exterior differential algebra to a standard form in which the only differential symbols are differentials of the underlying variables ($[\mathbf{dx}]$, $[\mathbf{dy}]$, etc.), and these symbols are always multiplied in the same order.

One key consequence of the anticommutativity relation is that $[\mathbf{df}]^2 = 0$, for all functions f. This is proved in exactly the same way we proved $\mathbf{v} \times \mathbf{v} = 0$ for all vectors $\mathbf{v}$: You can do it!

Example 5.

(a) Prove the formula $[\mathbf{df}]^2 = 0$.

(b) Let f and g be as in Example 3. Simplify $[\mathbf{df}][\mathbf{dg}]$ as much as possible. (We can simplify more, now that we know how to multiply.)

(c) Let F and G be as in Example 4. Simplify $[\mathbf{dF}][\mathbf{dG}]$ as much as possible.

(d) Let u and v be as in Example 1. Simplify $[\mathbf{du}][\mathbf{dv}]$ as much as possible.

(e) Let u and v be as in Example 2. Simplify $[\mathbf{du}][\mathbf{dv}]$ as much as possible.

3 Cheating our way to formulas from the book

Let r and θ be the polar coordinates, and ρ, θ, and ϕ be the spherical coordinates. The text instructs you to memorize the formulas $dydx = rdrd\theta$ and $dzdydx = \rho^2 \sin\phi d\rho d\theta d\phi$. These are good formulas, and you'll memorize them by accident if you do a lot of work in those coordinates, but the derivations in the text are difficult at best, so they feel like magic. Let's prove them.

Example 6. Let's derive the formula for polar conversion.

(a) Let $x = r\cos\theta$ and $y = r\sin\theta$. Rewrite $[\mathbf{dx}]$ and $[\mathbf{dy}]$ using only r and θ.

(b) Simplify $[\mathbf{dx}][\mathbf{dy}]$ as much as possible. (You will need the Pythagorean trig identity.)

(c) Simplify $[\mathbf{dy}][\mathbf{dx}]$ as much as possible. (You can repeat the same work, or use anticommutativity.)

(d) The answers to the previous two problems are not the same. Explain the difference in terms of the orientations of x and y, and of r and θ. (In two dimensions, "orientation" means "clockwise or counterclockwise".)

(e) Let's derive the formula in a more complicated way. Since $r^2 = x^2 + y^2$, we know $[\mathbf{d}(\mathbf{r^2})] = [\mathbf{d}(\mathbf{x^2} + \mathbf{y^2})]$. Simplify this to get a relationship between $[\mathbf{dr}]$, $[\mathbf{dx}]$, and $[\mathbf{dy}]$. Then do the same thing with $\tan\theta = \frac{y}{x}$, and bang the results together.

Example 7. I like to use the phrase "double polar coordinates" when talking about spherical coordinates. Here's why.

(a) Observe that $z = \rho\cos\phi$ and $r = r\sin\phi$. Use this and your work in Example 6 to show that $[\mathbf{dz}][\mathbf{dr}] = \rho[\mathbf{d}\rho][\mathbf{d}\phi]$.

(b) Use your work above to show that $[\mathbf{dx}][\mathbf{dy}][\mathbf{dz}] = r\rho[\mathbf{d}\rho][\mathbf{d}\phi][\mathbf{d}\theta]$.

(c) Substitute for r in the above to get the formula in the book.

(d) We can also derive the formula more gloriously without using the double polar insight. Use the formulas $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$ to express $[\mathbf{dx}]$, $[\mathbf{dy}]$, and $[\mathbf{dz}]$ in terms of only ρ, θ, and ϕ.

(e) Multiply out the three expressions in the previous problem, and simplify. (Actually, no, don't. It will take you forever. But think through what's likely to happen, and convince yourself that you could do it if it were important.)

4 The Jacobian alternative

We can use the techniques in this handout instead of the content of Chapter 15.6. In general I won't care which you use — you should prefer whatever you're more comfortable with. But I ask that you use no Jacobians on this handout, and that you use Jacobians on at least a few of the WebAssign problems from 15.6.

5 Using differential forms for integration

The examples here are translations of the examples in section 15.6. I find that chapter very difficult to understand as written, so reading these examples together may help make sense of that chapter.

Example 8. (Rogawski, Example 15.6.6.) Calculate $\displaystyle\iint_{\mathcal{P}} e^{4x-y}\,dxdy$, where $\mathcal{P}$ is the parallelogram with one vertex at the origin, and spanned by vectors $\langle 4,1\rangle$ and $\langle 3,3\rangle$.

(a) Find equations for the sides of $\mathcal{P}$.

(b) Let $u = x - y$ and $v = 4y - x$. Find formulas for the sides of $\mathcal{P}$ involving u and v.

(c) Find the relationship between $[\mathbf{du}][\mathbf{dv}]$ and $[\mathbf{dx}][\mathbf{dy}]$. (There are two ways to attack this. You can differentiate to express $[\mathbf{du}]$ and $[\mathbf{dv}]$ in terms of x, y, $[\mathbf{dx}]$, and $[\mathbf{dy}]$, or you can solve for x and y in terms of u and v, then get $[\mathbf{dx}]$ and $[\mathbf{dy}]$ in terms of u, v, $[\mathbf{du}]$, and $[\mathbf{dv}]$. In general I find that the first approach is a little easier, but either can work better depending on the problem.)

(d) Rewrite "$\displaystyle\iint_{\mathcal{P}}$" as an iterated integral using u and v.

(e) Rewrite "$e^{4x-y}dxdy$" using only u and v. (Unfortunately you have to solve for x and y in terms of u and v. If that had been a $4y - x$ you'd get out of it.)

(f) Evaluate the rewritten integral.

Example 9. (Rogawski, Example 15.6.7) Compute $\iint\limits_{\mathcal{D}} (x^2+y^2)dxdy$, where $\mathcal{D}$ is the domain $1 \le xy \le 4$, $1 \le \frac{y}{x} \le 4$ in the first quadrant.

(a) Let $u = xy$ and $v = \frac{y}{x}$. Express the boundaries of $\mathcal{D}$ in terms of u and v.

(b) Find the relationship between $[\mathbf{du}][\mathbf{dv}]$ and $[\mathbf{dx}][\mathbf{dy}]$.

(c) Rewrite "$\iint\limits_{\mathcal{D}}$" as an iterated integral using only u and v.

(d) Rewrite "$(x^2 + y^2)dxdy$" using only u and v. (I think you'll have to solve for x and y this time too.)

(e) Evaluate the rewritten integral.

Example 10. (Rogawski, Example 15.6.8) Integrate $f(x, y) = xy(x^2 + y^2)$ over $\mathcal{D}$, where $\mathcal{D}$ is the region in the first quadrant defined by the inequalities $-3 \le x^2 - y^2 \le 3$ and $1 \le xy \le 4$.

(a) Let $u = x^2 - y^2$ and $v = xy$. Express the boundaries of $\mathcal{D}$ in terms of u and v.

(b) Find the relationship between $[\mathbf{du}][\mathbf{dv}]$ and $[\mathbf{dx}][\mathbf{dy}]$.

(c) Rewrite "$\iint\limits_{\mathcal{D}}$" as an iterated integral using only u and v.

(d) Rewrite "$xy(x^2 + y^2)dxdy$" using only u and v. (This time it looks like you'll need to solve for x and y, but a miracle occurs and you get to avoid it.)

(e) Evaluate the rewritten integral.

6 Problems to turn in

Write up these problems carefully as a group, and don't put your names on the solutions until you're satisfied that you all understand and agree with them. You should focus on explaining your work as clearly as possible. When I grade, I will focus on clarity at least as much as on numerical correctness.

The only rules are that **you must acknowledge anyone (or any source)** outside your group that helps you, and **you may not use the Jacobians from Chapter 15.6** on these problems.

1. Simplify $(x^2 - y)[\mathbf{d(16 - x^2 - 16y^2)}]$ to an expression in which the only differential symbols are $[\mathbf{dx}]$ and $[\mathbf{dy}]$.

2. Fix Example 1. That is, express the area of the square (correctly) as a single iterated integral in variables u and v, and evaluate the integral.

3. Fix Example 2. That is, express the area of the parallelogram (correctly) as a single iterated integral in variables u and v, and evaluate the integral.

4. Let D be the region defined by the inequalities $10 \leq xy \leq 20$ and $20 \leq x^2 y \leq 40$. Rewrite $\iint\limits_D e^{xy}\,dxdy$ in terms of suitable coordinates u and v, then compute the integral. [The choice of u and v is not meant to be difficult. Draw a picture of the region D (don't draw it to scale) and label its boundary curves. There should be an obvious choice for u and v; don't be afraid to try it.]

5. **(Extra Credit)** Considering anticommutativity, why does Fubini's theorem say
$$\int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y)dydx = \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)dxdy$$
instead of
$$\int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y)dydx = -\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)dxdy?$$
Think about orientation, signed area, and handedness. [Hint?: What is $\int_{x=1}^{x=0} dx$? Now, let C_1 be the curve $y = x^2$ and C_2 be the curve $x = y^2$. Translate the double integrals $\int_{(0,0)}^{(1,1)} \int_{C_1}^{C_2} dydx$ and $\int_{(0,0)}^{(1,1)} \int_{C_1}^{C_2} dxdy$ into more normal notation as slowly and pedantically as possible.]