Math 2163

Jeff Mermin's section, Final Exam, December 6

On the essay questions (# 2–10) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

Put away your calculators, cell phones, smart watches, and other internet-enabled devices for the duration of the exam. If any of these are out for any reason, it will constitute an academic integrity violation.

1. (**50 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   In the statements below, a, b, c, and d are numbers, t is a parameter, x, y, and z are the usual rectangular coordinates, $f = f(x, y)$ is a function, $g = g(x, y)$, $h = h(x, y)$, and $F = F(x, y, z)$ are smoothly differentiable functions, $\mathbf{v}$ and $\mathbf{w}$ are vectors, $\mathbf{r} = \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a smooth parametric curve, and D is a closed, bounded, simply connected region inside $\mathbb{R}^2$.

   (a) $\mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{v}$.

   (b) $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

   (c) $\displaystyle\int_a^b \int_c^d f(x, y)dxdy = \int_a^b \int_c^d f(x, y)dydx.$

   (d) $\nabla(g + h) = \nabla g + \nabla h$.

   (e) If g has an absolute maximum on the region D at (a, b), then g has a local maximum at (a, b).

   (f) If $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = L$, then $\displaystyle\lim_{x\to a} f(x, b) = L$.

   (g) $\nabla g(a, b)$ is tangent to the level curve of $g(x, y)$ passing through (a, b).

   (h) $\displaystyle\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right).$

   (i) $\displaystyle\int_{t=0}^{t=1} \frac{d\mathbf{r}}{dt}dt = \mathbf{r}(1) - \mathbf{r}(0).$

   (j) If $F(a, b, c) = 0$, then $\nabla F(a, b, c)$ is normal to the surface $F(x, y, z)$ at the point (a, b, c).

2. (**20 points**) Let $\mathbf{v} = \langle 3, -1, 4 \rangle$ and $\mathbf{w} = \langle 0, -2, 3 \rangle$.

   (a) Compute $\mathbf{v} \bullet \mathbf{w}$.

   (b) Compute $\mathbf{v} \times \mathbf{w}$.

3. (**15 points**) Let ℓ be the line with parametric equation $(x, y, z) = (3, 1, 5) + \langle -2, 5, 4 \rangle t$, and let F be the plane with equation $2x - y - 3z = 11$. Do ℓ and F intersect? If they do, find the point of intersection. If they don't, explain (in a sentence or two, backed up by a computation if necessary) why not.

4. (**15 points**) Find an equation for the tangent line to the parametric curve $\mathbf{r}(t) = (e^t, e^{2t}, t^2)$ at the point $(1, 1, 0)$.

5. (**20 points**) Find an equation for the tangent plane to the graph of the equation $z = x^2y^3$ at the point $(3, 2, 72)$.

6. (**20 points**) Find the directional derivative $D_{\mathbf{u}}f(P)$ if $f(x, y, z) = xe^{yz}$, $P = (3, 0, 5)$, and $\mathbf{u} = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle$. (You may assume that $\mathbf{u}$ is a unit vector.)

7. (**15 points**) Consider the function $f(x, y) = \tan^{-1} xy$. You may assume that

$$f_x = \frac{y}{1 + x^2y^2} \qquad f_y = \frac{x}{1 + x^2y^2}$$

$$f_{xx} = -\frac{2xy^3}{(1 + x^2y^2)^2} \qquad f_{xy} = \frac{1 - x^2y^2}{(1 + x^2y^2)^2} \qquad f_{yy} = -\frac{2x^3y}{(1 + x^2y^2)^2}$$

Exactly one of the points $P = (0, 0)$, $Q = (1, 1)$ is a critical point of f. Use appropriate computations to verify that this statement is true, then determine whether the critical point is a local maximum, local minimum, or saddle point.

8. (**50 points**) Express the following quantities using iterated integrals. You may use any coordinate system, provided you use it correctly. If you use coordinates other than rectangular, polar, cylindrical, or spherical, be sure to define them. **Do not evaluate the integrals**.

   (a) The volume of the region E inside the ellipsoid $4x^2 + 9y^2 + z^2 = 36$.

   (b) $\iint_D (x^2+y^2)dydx$, where D is the region in the first quadrant bounded by the curves $x^2 - y^2 = 1$, $y^2 - x^2 = 1$, and $xy = 1$.
   [Hint: the x and y axes, taken together, have equation $xy = 0$.]

9. (**50 points**) Let $F(x, y) = \langle y, x\rangle$ and $G(x, y) = \langle y, -x\rangle$. Let C be the path from $(1, 0)$ to $(0, 1)$ that proceeds in a vertical line segment from $(1, 0)$ to $(1, 1)$ and then in a horizontal line segment from $(1, 1)$ to $(0, 1)$.

   (a) Exactly one of F and G is conservative. Use an appropriate computation or computations to identify the conservative vector field and to verify that the other is not conservative.

   (b) Compute $\displaystyle\int_C F \bullet d\mathbf{r}$, using any appropriate technique.

   (c) Compute $\displaystyle\int_C G \bullet d\mathbf{r}$, using any appropriate technique.

10. (**Extra credit: 10 points**) Write down an iterated integral which expresses the surface area of the parabolic dome $z = 1 - x^2 - y^2$, $z \geq 0$. **Do not evaluate**.

11. (**Extra credit: 10 points**) In the space remaining on this page, write the largest integer you can. You do not have to simplify, as, for example, "10^3" may take up less space than "1000". However, I should be able (at least in theory) to determine the precise value of your number with no ambiguity or reference to the experiential universe (so, for example, "the number of stars in the sky" is right out). You may need to use some space defining your notation.