Math 2163

Jeff Mermin's section, Final Exam, December 9

On the essay questions (# 2–10) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

Put away your calculators, cell phones, smart watches, and other internet-enabled devices for the duration of the exam. If any of these are out for any reason, it will constitute an academic integrity violation.

1. (**50 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   In the statements below, a and b are numbers, x, y, and z are the usual rectangular coordinates for $\mathbb{R}^2$, r and θ are the usual polar coordinates, t is a parameter, $\mathbf{r}$ and $\mathbf{s}$ are parametric curves, f, P, and Q are smoothly differentiable functions, R is a simply connected region, and $F = \langle P, Q\rangle$ is a smoothly differentiable vector field on R,

   (a) $\displaystyle\iint_R f\,dxdy = \iint_R f\,drd\theta$

   (b) There are eight possible orders of integration for a triple iterated integral.

   (c) There is a function $g(x,y)$ such that $g_x = x^2 + y^2$ and $g_y = x^2 - y^2$.

   (d) If $\mathbf{v}$ is a direction vector for a line ℓ, then $2\mathbf{v}$ is also a direction vector for ℓ.

   (e) The vector $\langle 2,3,4\rangle$ is parallel to the plane with equation $2x+3y+4z = 5$.

   (f) $\displaystyle\frac{d(\mathbf{r}\bullet\mathbf{s})}{dt} = \frac{d\mathbf{r}}{dt}\bullet\frac{d\mathbf{s}}{dt}.$

   (g) $\displaystyle\iiint_R f dxdydz = \iiint_R f r dr d\theta dz$

   (h) If $F(x,y) = \langle P(x,y), Q(x,y)\rangle$ is conservative, then $P_y = Q_x$.

   (i) Two lines in $\mathbf{R}^3$ define a plane.

   (j) $\nabla f(a,b)$ is normal to the level curve of $f(x,y)$ passing through (a,b).

2. (**20 points**) Let $\mathbf{v} = \langle 4, 5, -2 \rangle$ and $\mathbf{w} = \langle 5, -4, 0 \rangle$.

   (a) Compute $\mathbf{v} \bullet \mathbf{w}$.

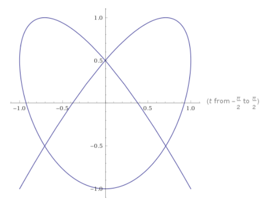
   (b) Compute $\mathbf{v} \times \mathbf{w}$.

3. (**15 points**) Let $P = (3, 1, 4)$ and $Q = (4, -1, -2)$.

   (a) Find a parametric equation for the line through P and Q.

   (b) Find an equation for the plane consisting of all points equidistant from P and Q. (That is, a point R is on this plane if and only if the distance from R to P is equal to the distance from R to Q.)

4. (**40 points**) The parametric curve $C : (x, y) = (\sin 3t, -\cos 4t)$ (for $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$) traces out the pretzel shape shown below.



(a) C passes through the point $(0, \frac{1}{2})$ twice, so there are two different tangent lines. Find the equation of one of them.

(b) Compute $\displaystyle\int_C y\,dx + x\,dy$, using any correct method. (You should not need the integral tables, but they're attached at the end of the exam just in case.)

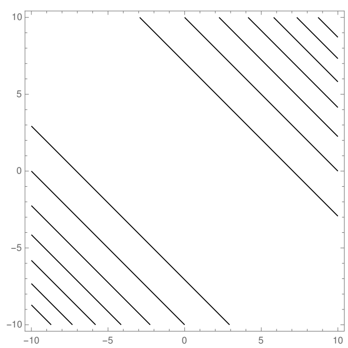
5. (**20 points**) Match the functions with their contour plots. No justification is necessary for correct answers, but incorrect answers with good thinking behind them may earn partial credit.

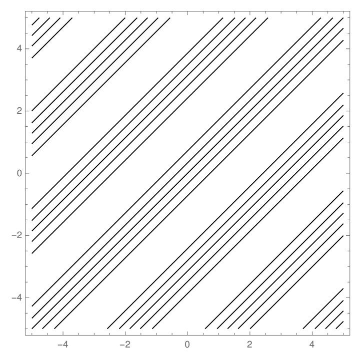
   (a) $f(x, y) = \sin(x - y)$

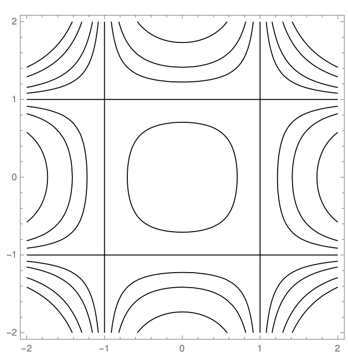
   (b) $g(x, y) = (x - 1)(x + 1)(y - 1)(y + 1)$

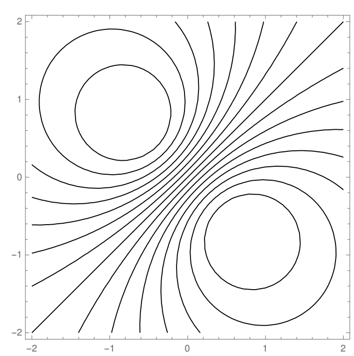
   (c) $h(x, y) = \dfrac{x - y}{1 + x^2 + y^2}$.

   (d) $k(x, y) = (x + y)^2$.









6. (**25 points**) Consider the function $z = f(x, y) = x^3y + 12x^2 - 8y$.

(a) f has one critical point. Find it.

(b) Determine, with justification, whether the critical point is a local minimum, a local maximum, or a saddle point.

7. (**60 points**) Express the following quantities using iterated integrals or single integrals. You may use any coordinate system, provided you use it correctly. If you use coordinates other than rectangular, polar, cylindrical, or spherical, be sure to define them. **Do not evaluate the integrals**.

   (a) The volume of the region E bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $z = 0$, $x = 0$, and $x = 1$.

   (b) The mass of the region above the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 2$, if the density at the point x, y, z is given by $x^2 + z$.

(c) $\iint_P x\,dy\,dx$, where P is the parallelogram with vertices $(0,0)$, $(6,-2)$, $(18,1)$, and $(12,3)$.

8. (**20 points**) Let $F(x,y) = \langle x^2, y^2 \rangle$. Determine whether or not F is conservative. If it is conservative, find a potential function.

9. (**Extra credit: 10 points**) The *standard normal density* or *bell curve* is very important in probability and statistics, where it is used to describe certain kinds of repeatable experiments. The bell curve is given by the function $f(x) = \frac{1}{A}e^{-\frac{x^2}{2}}$, where A is chosen to make the bell curve into a probability density, that is, $\displaystyle\int_{x=-\infty}^{x=\infty} f(x)\,dx = 1$.

   Thus $A = \displaystyle\int_{x=-\infty}^{x=\infty} e^{-\frac{x^2}{2}}\,dx$. Unfortunately, it is known that the indefinite integral $\displaystyle\int e^{-\frac{x^2}{2}}\,dx$ cannot be evaluated algebraically. However, we can still solve for the exact value of A.

   Let $B = \displaystyle\int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty} e^{-\frac{x^2}{2}}e^{-\frac{y^2}{2}}\,dydx$.

   (a) Describe the relationship between A and B.

   (b) Rewrite B using polar coordinates.

   (c) Find A.

10. (**Extra credit: 10 points**) In the space remaining on this page, write the largest integer you can. You do not have to simplify, as, for example, "10^3" may take up less space than "1000". However, I should be able (at least in theory) to determine the precise value of your number with no ambiguity or reference to the experiential universe (so, for example, "the number of stars in the sky" is right out). You may need to use some space defining your notation.