Math 2163

Jeff Mermin's section, Final Exam, December 15

On the essay questions (# 2–12) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

You won't need the integral tables on the following page, but they're available anyway.

1. (**50 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   In the statements below, c is a number, x, y, and z are the usual rectangular coordinates, $f = f(x, y, z)$, $g = g(x, y, z)$, and $h = h(x, y)$ are smoothly differentiable functions, t is a parameter, $\mathbf{r} = \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a parametric curve, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors, D is a region in $\mathbb{R}^3$, F is a vector field on $\mathbb{R}^3$, and R is a region in $\mathbb{R}^2$.

   (a) If $f(x, y, z) \geq g(x, y, z)$ for all $(x, y, z) \in D$, then
   $$\iiint_D f dV \geq \iiint_D g dV.$$

   (b) $|\mathbf{v} \bullet \mathbf{w}| = \|\mathbf{v}\|\|\mathbf{w}\|$.

   (c) $\displaystyle\iiint_D f(x, y, z) dxdydz = \iiint_D f(x, y, z) dzdxdy$.

   (d) If $F(x, y, z) = c\dfrac{\langle x, y, z\rangle}{|\langle x, y, z\rangle|^3}$ is an "inverse square law" field, then F is conservative.

   (e) $\nabla(fg) = \nabla f \times \nabla g$.

   (f) If $h(x, y) \geq 0$ for all $(x, y) \in R$, then $\displaystyle\iint_R h dA \geq 0$.

   (g) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$.

   (h) $\dfrac{d}{dt}|\mathbf{r}| = \left|\dfrac{d\mathbf{r}}{dt}\right|$.

   (i) If D is closed and bounded, then f has a global maximum on D.

   (j) The area of R is $\displaystyle\iint_R 1 dA$.

2. (**15 points**) Let $\mathbf{v} = \langle 4, 3, 2 \rangle$ and $\mathbf{w} = \langle 2, 5, -1 \rangle$.

   (a) Compute $\mathbf{v} \bullet \mathbf{w}$.

   (b) Compute $\mathbf{v} \times \mathbf{w}$.

3. (**5 points**) Find three points on the plane $2x + 4y + z = 2$.

4. (**15 points**) Find an equation for the plane consisting of all points equidistant from $P = (5, 5, -2)$ and $Q = (0, 0, -2)$. (That is, a point R is on this plane if and only if the distance from R to P is equal to the distance from R to Q.)

5. (**10 points**) What does Green's Theorem say?

6. (**15 points**) Sketch an example of a region which is ...

(a) ... simply connected.

(b) ... connected but not simply connected.

(c) ... not connected.

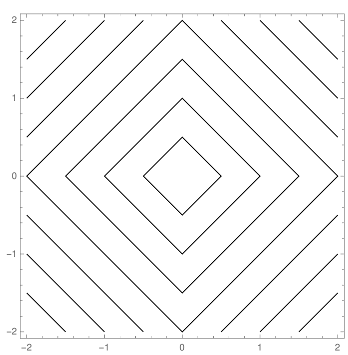
7. (**20 points**) Match the functions with their contour plots. No justification is necessary for correct answers, but incorrect answers with good thinking behind them may earn partial credit.

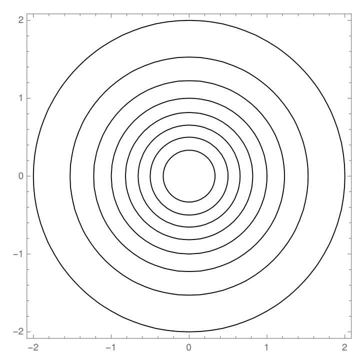
(a) $f(x,y) = \dfrac{1}{1+x^2+y^2}$.

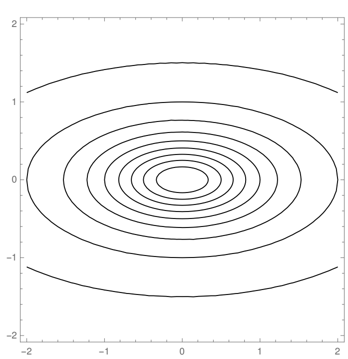
(b) $g(x,y) = \dfrac{\cos(y)}{e^{\sqrt{x^2+y^2}}}$.

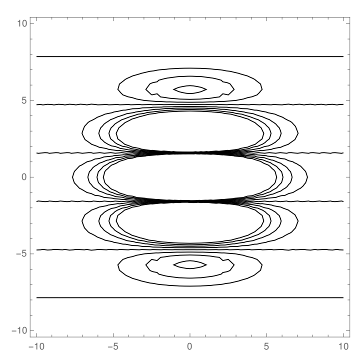
(c) $h(x,y) = \dfrac{1}{1+x^2+4y^2}$.

(d) $k(x,y) = |x| + |y|$.









8. (**15 points**) Find all critical points of the function $f(x,y) = x^3 + y^3 - xy$.

9. (**60 points**) Express the following quantities using iterated integrals or single integrals. You may use any coordinate system, provided you use it correctly. If you use coordinates other than rectangular, polar cylindrical, or spherical, be sure to define them. **Do not evaluate the iterated integrals**.

   (a) $\iiint_W z\,dx\,dy\,dz$, where W is the region between the xy-plane and the paraboloid $z = x^2 + y^2$, and inside the cylinder $x^2 + y^2 = 9$.

(b) $\iiint_R (x^2 + y^2)\,dx\,dy\,dz$, where R is the region inside the sphere $x^2 + y^2 + z^2 = 1$, and below the xy-plane.

(c) $\iint_P (5y - x)(4x - 3y)\,dy\,dx$, where P is the parallelogram with vertices $(0, 0)$, $(5, 1)$, $(8, 5)$, and $(3, 4)$.

10. (**45 points**) Let $f(x, y, z) = x \ln(1 + y^2 + z^2)$, and let $F = \nabla f$.

(a) Compute $F(0, 0, 0)$ and $F(1, 1, 1)$.

(b) Is F conservative? Explain how you know.

(c) Let C be the quarter-turn of the helix $(x, y, z) = (\cos t, \sin t, t)$ from $(1, 0, 0)$ to $(0, 1, \frac{\pi}{2})$. Compute $\int_C F \bullet \langle dx, dy, dz \rangle$.

11. (**Extra credit: 10 points**) Clearly describe the region T for which the triple integral $\iiint_T (1 - x^2 - y^2 - z^2)dxdydz$ has the maximum possible value.

12. (**Extra credit: 10 points**) In the space remaining on this page, write the largest integer you can. You do not have to simplify, as, for example, "10^3" may take up less space than "1000". However, I should be able (at least in theory) to determine the precise value of your number with no ambiguity or reference to the experiential universe (so, for example, "the number of stars in the sky" is right out). You may need to use some space defining your notation.