Math 2163

Jeff Mermin's section, Final exam, December 5

On the essay questions (# 2–13) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

Unless specifically instructed to do so, **you do not need to evaluate integrals** on this exam. If your answer is a double or triple integral, you should leave it as an iterated integral, after simplifying the integrand and limits of integration into a reasonable form. If your answer is a single integral, you should simplify it to a form that a calculus II student would understand (that is, $\int_{x=a}^{x=b} f(x)dx$ for some function f and numbers a and b, or similarly for y, z, t, etc. in place of x) with everything simplified into a reasonable form. (If you are instructed to evaluate, your answer should be a number.)

An answer is simplified into a reasonable form if a reasonable professional would not be offended to find it written that way in a problem description. So "$x^2 + 2x + 2$" and "$(x + 1)^2 + 1$" are both okay, as is "$\frac{2A+1}{\sqrt{A-1}}$, where $A = (x + 1)^2 + 1$". But "$x^2 + \frac{4x+2+2}{2}$" is not.

You probably won't find the integral tables on the following pages helpful, but they're there just in case.

Basic Forms

1. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

2. $\int \frac{du}{u} = \ln|u| + C$

3. $\int e^u\,du = e^u + C$

4. $\int a^u\,du = \frac{a^u}{\ln a} + C$

5. $\int \sin u\,du = -\cos u + C$

6. $\int \cos u\,du = \sin u + C$

7. $\int \sec^2 u\,du = \tan u + C$

8. $\int \csc^2 u\,du = -\cot u + C$

9. $\int \sec u \tan u\,du = \sec u + C$

10. $\int \csc u \cot u\,du = -\csc u + C$

11. $\int \tan u\,du = \ln|\sec u| + C$

12. $\int \cot u\,du = \ln|\sin u| + C$

13. $\int \sec u\,du = \ln|\sec u + \tan u| + C$

14. $\int \csc u\,du = \ln|\csc u - \cot u| + C$

15. $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$

16. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

Exponential and Logarithmic Forms

17. $\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au} + C$

18. $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$

19. $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

20. $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

21. $\int \ln u\,du = u\ln u - u + C$

22. $\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

23. $\int \frac{1}{u\ln u}\,du = \ln|\ln u| + C$

Hyperbolic Forms

24. $\int \sinh u\,du = \cosh u + C$

25. $\int \cosh u\,du = \sinh u + C$

26. $\int \tanh u\,du = \ln\cosh u + C$

27. $\int \coth u\,du = \ln|\sinh u| + C$

28. $\int \operatorname{sech} u\,du = \tan^{-1}|\sinh u| + C$

29. $\int \operatorname{csch} u\,du = \ln\left|\tanh\frac{1}{2}u\right| + C$

30. $\int \operatorname{sech}^2 u\,du = \tanh u + C$

31. $\int \operatorname{csch}^2 u\,du = -\coth u + C$

32. $\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$

33. $\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$

Trigonometric Forms

34. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

35. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

36. $\int \tan^2 u\,du = \tan u - u + C$

37. $\int \cot^2 u\,du = -\cot u - u + C$

38. $\int \sin^3 u\,du = -\frac{1}{3}(2+\sin^2 u)\cos u + C$

39. $\int \cos^3 u\,du = \frac{1}{3}(2+\cos^2 u)\sin u + C$

40. $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

41. $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

42. $\int \sec^3 u\,du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

$$\int \csc^3 u\,du = -\frac{1}{n}\csc u \cot u + \frac{1}{n}\ln|\csc u - \cot u| + C$$

$$\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1}u\cos u + \frac{n-1}{n}\int \sin^{n-2}u\,du$$

$$\int \cos^n u\,du = \frac{1}{n}\cos^{n-1}u\sin u + \frac{n-1}{n}\int \cos^{n-2}u\,du$$

$$\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1}u - \int \tan^{n-2}u\,du$$

$$\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1}u - \int \cot^{n-2}u\,du$$

$$\int \sec^n u\,du = \frac{1}{n-1}\tan u \sec^{n-2}u + \frac{n-2}{n-1}\int \sec^{n-2}u\,du$$

$$\int \csc^n u\,du = \frac{-1}{n-1}\cot u \csc^{n-2}u + \frac{n-2}{n-1}\int \csc^{n-2}u\,du$$

$$\int \sin au \sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$$

$$\int \cos au \cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$$

$$\int \sin au \cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$$

$$\int u\sin u\,du = \sin u - u\cos u + C$$

$$\int u\cos u\,du = \cos u + u\sin u + C$$

$$\int u^n \sin u\,du = -u^n\cos u + n\int u^{n-1}\cos u\,du$$

$$\int u^n \cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$$

$$\int \sin^n u\cos^m u\,du$$
$$= -\frac{\sin^{n-1}u\cos^{m+1}u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2}u\cos^m u\,du$$
$$= \frac{\sin^{n+1}u\cos^{m-1}u}{n+m} + \frac{m-1}{n+m}\int \sin^n u\cos^{m-2}u\,du$$

rse Trigonometric Forms

$$\int \sin^{-1} u\, du = u \sin^{-1} u + \sqrt{1-u^2} + C$$

$$\int \cos^{-1} u\, du = u \cos^{-1} u - \sqrt{1-u^2} + C$$

$$\int \tan^{-1} u\, du = u \tan^{-1} u - \frac{1}{2}\ln(1+u^2) + C$$

$$\int u \sin^{-1} u\, du = \frac{2u^2-1}{4}\sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$$

$$\int u \cos^{-1} u\, du = \frac{2u^2-1}{4}\cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$$

$$\int u \tan^{-1} u\, du = \frac{u^2+1}{2}\tan^{-1} u - \frac{u}{2} + C$$

$$\int u^n \sin^{-1} u\, du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1} u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$$

$$\int u^n \cos^{-1} u\, du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1} u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$$

$$\int u^n \tan^{-1} u\, du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1} u - \int \frac{u^{n+1}\,du}{1+u^2}\right], \quad n \neq -1$$

Forms Involving $\sqrt{a^2-u^2}$, $a > 0$

67. $$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

68. $$\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$$

69. $$\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$$

70. $$\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2-u^2} - \sin^{-1}\frac{u}{a} + C$$

71. $$\int \frac{u^2\,du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

72. $$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$$

73. $$\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{1}{a^2u}\sqrt{a^2-u^2} + C$$

74. $$\int (a^2-u^2)^{3/2}\,du = -\frac{u}{8}(2u^2-5a^2)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$$

75. $$\int \frac{du}{(a^2-u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2-u^2}} + C$$

Forms Involving $\sqrt{u^2-a^2}$, $a > 0$

76. $$\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$$

77. $$\int u^2\sqrt{u^2-a^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$$

78. $$\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\cos^{-1}\frac{a}{|u|} + C$$

79. $$\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$$

80. $$\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$$

81. $$\int \frac{u^2\,du}{\sqrt{u^2-a^2}} = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$$

82. $$\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C$$

83. $$\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$$

Forms Involving $\sqrt{a^2+u^2}$, $a > 0$

84. $$\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$$

85. $$\int u^2\sqrt{a^2+u^2}\,du = \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right) + C$$

86. $$\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$$

87. $$\int \frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u+\sqrt{a^2+u^2}\right) + C$$

88. $$\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u+\sqrt{a^2+u^2}\right) + C$$

89. $$\int \frac{u^2\,du}{\sqrt{a^2+u^2}} = \frac{u}{2}\sqrt{a^2+u^2} - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$$

90. $$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$$

91. $$\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$$

92. $$\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$$

Forms Involving $a + bu$

93. $$\int \frac{u\,du}{a+bu} = \frac{1}{b^2}\left(a+bu-a\ln|a+bu|\right) + C$$

94. $$\int \frac{u^2\,du}{a+bu} = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$$

95. $$\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$$

96. $$\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

97. $$\int \frac{u\,du}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$$

98. $$\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

99. $$\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left(a+bu-\frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$$

100. $$\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2} + C$$

101. $$\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$$

102. $$\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$$

103. $$\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a+bu}}$$

104. $$\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, \quad \text{if } a > 0$$
$$= \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}} + C, \quad \text{if } a < 0$$

105. $$\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$$

106. $$\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$$

107. $$\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$$

Forms Involving $\sqrt{2au-u^2}$, $a > 0$

108. $$\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$$

109. $$\int u\sqrt{2au-u^2}\,du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$$

110. $$\int \frac{du}{\sqrt{2au-u^2}} = \cos^{-1}\left(\frac{a-u}{a}\right) + C$$

111. $$\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C$$

1. (**50 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   In the statements below, x, y, z, r, and θ are the usual rectangular and polar coordinates, f is a smoothly differentiable function, L is a real number R is a closed and bounded region, and **a**, **b**, and **c** are vectors. All partial derivatives that appear in these statements may be assumed to exist, and everything appearing in a denominator may be assumed to be nonzero.

   (a) If $R = \{(x, y, z) : 1 \le x^2 + y^2 + z^2 \le 16\}$, then $\iiint\limits_R dV = 84\pi$.

   (b) The vector $\langle 2, 3, 4\rangle$ is parallel to the plane with equation $2x+3y+4z = 5$.

   (c) If $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$, then $\lim\limits_{x\to a} f(x, b) = L$.

   (d) If z is defined implicitly as a function of x and y by $f(x, y, z) = 0$, then
   $$\frac{\partial z}{\partial x} = \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}.$$

   (e) The vector $\langle dx, dy, dz\rangle$ is normal to the graph of $z = f(x, y)$.

(f) $\mathbf{a}$ and $-2\mathbf{a}$ point in the same direction.

(g) $(\mathbf{a} \bullet \mathbf{b})\mathbf{c} = \mathbf{a}(\mathbf{b} \bullet \mathbf{c})$.

(h) If f has a local maximum at $(0,0)$, then $f_x(0,0) = 0$.

(i) $\frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$.

(j) $\iint\limits_R f \, dr d\theta = \iint\limits_R f \, d\theta dr$.

2. (**20 points**) Let $\mathbf{x} = \langle 3, 3, 4 \rangle$ and $\mathbf{y} = \langle 2, 1, 3 \rangle$.

   (a) Compute $\mathbf{x} \bullet \mathbf{y}$.

   (b) Compute $\mathbf{x} \times \mathbf{y}$.

3. (**20 points**) Find a parametric equation of the line through the point $P = (-3, 1, -1)$ which is normal to the plane $F : 4x - 3y - 3z = -1$.

4. (**20 points**) Find an equation for the tangent plane to the quadric surface $4x^2 + 2y^2 + 4z^2 + 2xz + 3yz = 85$ at the point $(2, -3, -3)$.

5. (**20 points**) Compute $\dfrac{dy}{dx}$, assuming

$$y = \left(\frac{2x+5}{3x-2}\right)^3 - 4\left(\frac{2x+5}{3x-2}\right)^2 \left(x^2 \cos x + \ln x\right) + 6\left(\frac{2x+5}{3x-2}\right)\left(x^2 \cos x + \ln x\right)^2 - 4\left(x^2 \cos x + \ln x\right)^3.$$

6. (**20 points**) Find all the critical points for the function $z(x, y) = x^3 + 2xy - 2y^2 - 10x$.

7. (**20 points**) Sketch the contour map of a function with saddle points at $(0, 0)$ and $(2, 0)$ and a local minimum at $(2, 1)$. Clearly label these and any other critical points.

8. (**20 points**) Evaluate the double integral $\int_{x=0}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} x dy dx$. You may use any correct method, but if you do something other than direct computation (e.g., change the order of integration, or change coordinate systems, or invoke a theorem) make it absolutely clear what you're doing.

9. (**20 points**) A object occupies the region in the first octant bounded by the surfaces $x^2+y^2 = 4$ and $z = 2y$. If its density is given by density$(x, y, z) = z^2 + 1$, express its mass as an iterated integral. **Do not evaluate**.

10. **(20 points)** Let S be the "topologist's spring", parametrized by $(x, y, z) = (\cos t, \sin t, e^t)$, and let C be the arc of S going from $P = \left(-1, 0, \frac{1}{e^{\pi}}\right)$ to $Q = (0, 1, e^{\frac{\pi}{2}})$.

(a) Find the length of C. **Do not evaluate**.

(b) Evaluate $\int_C F \bullet d\mathbf{r}$, where F is the vector field $\langle yz, xz, xy \rangle$.

11. **(20 points)**

(a) Give an example of a conservative vector field on $\mathbb{R}^2$ or $\mathbb{R}^3$, and explain how you know it's conservative.

(b) Give an example of a non-conservative vector field on $\mathbb{R}^2$ or $\mathbb{R}^3$, and explain how you know it isn't conservative.

12. (**Extra credit: 10 points**) In the space remaining on this page, write the largest integer you can. You do not have to simplify, as, for example, "10^3" may take up less space than "1000". However, I should be able (at least in theory) to determine the precise value of your number with no ambiguity or reference to the experiential universe (so, for example, "the number of stars in the sky" is right out), so you may need to use some space defining your notation.

13. (**Extra credit: 10 points**) Prove or disprove the following statement:

    Suppose that $F(x, y)$ is a smoothly differentiable function on all of $\mathbb{R}^2$. Then, if $f = F_{xy}$ is its mixed second partial derivative, and $a < b$ and $c < d$ are numbers, we have

$$\int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x, y) dy dx = F(b, d) + F(a, c) - F(b, c) - F(a, d).$$