 Name:__________________________

Math 2163

Jeff Mermin's section, Test 3, November 18

On the essay questions (# 2–6) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

You may or may not find the integral tables on the following page helpful.

Basic Forms

1. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C,\quad n \neq -1$
2. $\int \frac{du}{u} = \ln|u| + C$
3. $\int e^u\,du = e^u + C$
4. $\int a^u\,du = \frac{a^u}{\ln a} + C$
5. $\int \sin u\,du = -\cos u + C$
6. $\int \cos u\,du = \sin u + C$
7. $\int \sec^2 u\,du = \tan u + C$
8. $\int \csc^2 u\,du = -\cot u + C$
9. $\int \sec u \tan u\,du = \sec u + C$
10. $\int \csc u \cot u\,du = -\csc u + C$
11. $\int \tan u\,du = \ln|\sec u| + C$
12. $\int \cot u\,du = \ln|\sin u| + C$
13. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
14. $\int \csc u\,du = \ln|\csc u - \cot u| + C$
15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$
16. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

Exponential and Logarithmic Forms

17. $\int ue^{au}\,du = \frac{1}{a^2}(au - 1)e^{au} + C$
18. $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$
19. $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2 + b^2}(a \sin bu - b \cos bu) + C$
20. $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2 + b^2}(a \cos bu + b \sin bu) + C$
21. $\int \ln u\,du = u \ln u - u + C$
22. $\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$
23. $\int \frac{1}{u \ln u}\,du = \ln|\ln u| + C$

Hyperbolic Forms

24. $\int \sinh u\,du = \cosh u + C$
25. $\int \cosh u\,du = \sinh u + C$
26. $\int \tanh u\,du = \ln \cosh u + C$
27. $\int \coth u\,du = \ln|\sinh u| + C$
28. $\int \operatorname{sech} u\,du = \tan^{-1}|\sinh u| + C$
29. $\int \operatorname{csch} u\,du = \ln\left|\tanh \frac{1}{2}u\right| + C$
30. $\int \operatorname{sech}^2 u\,du = \tanh u + C$
31. $\int \operatorname{csch}^2 u\,du = -\coth u + C$
32. $\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$
33. $\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$

Trigonometric Forms

34. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$
35. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$
36. $\int \tan^2 u\,du = \tan u - u + C$
37. $\int \cot^2 u\,du = -\cot u - u + C$
38. $\int \sin^3 u\,du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$
39. $\int \cos^3 u\,du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$
40. $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$
41. $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$
42. $\int \sec^3 u\,du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$
43. $\int \csc^3 u\,du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$
44. $\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$
45. $\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$
46. $\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$
47. $\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\,du$
48. $\int \sec^n u\,du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$
49. $\int \csc^n u\,du = \frac{-1}{n-1}\cot u \csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\,du$
50. $\int \sin au \sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$
51. $\int \cos au \cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$
52. $\int \sin au \cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$
53. $\int u \sin u\,du = \sin u - u \cos u + C$
54. $\int u \cos u\,du = \cos u + u \sin u + C$
55. $\int u^n \sin u\,du = -u^n \cos u + n\int u^{n-1}\cos u\,du$
56. $\int u^n \cos u\,du = u^n \sin u - n\int u^{n-1}\sin u\,du$
57. $\int \sin^n u \cos^m u\,du$
$= -\frac{\sin^{n-1} u \cos^{m+1} u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} u \cos^m u\,du$
$= \frac{\sin^{n+1} u \cos^{m-1} u}{n+m} + \frac{m-1}{n+m}\int \sin^n u \cos^{m-2} u\,du$

Inverse Trigonometric Forms

58. $\int \sin^{-1} u\,du = u \sin^{-1} u + \sqrt{1 - u^2} + C$
59. $\int \cos^{-1} u\,du = u \cos^{-1} u - \sqrt{1 - u^2} + C$
60. $\int \tan^{-1} u\,du = u \tan^{-1} u - \frac{1}{2}\ln(1 + u^2) + C$
61. $\int u \sin^{-1} u\,du = \frac{2u^2 - 1}{4}\sin^{-1} u + \frac{u\sqrt{1 - u^2}}{4} + C$
62. $\int u \cos^{-1} u\,du = \frac{2u^2 - 1}{4}\cos^{-1} u - \frac{u\sqrt{1 - u^2}}{4} + C$
63. $\int u \tan^{-1} u\,du = \frac{u^2 + 1}{2}\tan^{-1} u - \frac{u}{2} + C$
64. $\int u^n \sin^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1} u - \int \frac{u^{n+1}\,du}{\sqrt{1 - u^2}}\right],\quad n \neq -1$
65. $\int u^n \cos^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1} u + \int \frac{u^{n+1}\,du}{\sqrt{1 - u^2}}\right],\quad n \neq -1$
66. $\int u^n \tan^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1} u - \int \frac{u^{n+1}\,du}{1 + u^2}\right],\quad n \neq -1$

Forms Involving $\sqrt{a^2 - u^2}$, $a > 0$

67. $\int \sqrt{a^2 - u^2}\,du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$
68. $\int u^2\sqrt{a^2 - u^2}\,du = \frac{u}{8}(2u^2 - a^2)\sqrt{a^2 - u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$
69. $\int \frac{\sqrt{a^2 - u^2}}{u}\,du = \sqrt{a^2 - u^2} - a \ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$
70. $\int \frac{\sqrt{a^2 - u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2 - u^2} - \sin^{-1}\frac{u}{a} + C$
71. $\int \frac{u^2\,du}{\sqrt{a^2 - u^2}} = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$
72. $\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$
73. $\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{1}{a^2 u}\sqrt{a^2 - u^2} + C$
74. $\int (a^2 - u^2)^{3/2}\,du = -\frac{u}{8}(2u^2 - 5a^2)\sqrt{a^2 - u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} +$
75. $\int \frac{du}{(a^2 - u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2 - u^2}} + C$

Forms Involving $\sqrt{u^2 - a^2}$, $a > 0$

76. $\int \sqrt{u^2 - a^2}\,du = \frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$
77. $\int u^2\sqrt{u^2 - a^2}\,du$
$= \frac{u}{8}(2u^2 - a^2)\sqrt{u^2 - a^2} - \frac{a^4}{8}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$
78. $\int \frac{\sqrt{u^2 - a^2}}{u}\,du = \sqrt{u^2 - a^2} - a \cos^{-1}\frac{a}{|u|} + C$
79. $\int \frac{\sqrt{u^2 - a^2}}{u^2}\,du = -\frac{\sqrt{u^2 - a^2}}{u} + \ln\left|u + \sqrt{u^2 - a^2}\right| + C$
80. $\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C$
81. $\int \frac{u^2\,du}{\sqrt{u^2 - a^2}} = \frac{u}{2}\sqrt{u^2 - a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$
82. $\int \frac{du}{u^2\sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u} + C$
83. $\int \frac{du}{(u^2 - a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2 - a^2}} + C$

Forms Involving $\sqrt{a^2 + u^2}$, $a > 0$

84. $\int \sqrt{a^2 + u^2}\,du = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2}\ln(u + \sqrt{a^2 + u^2}) + C$
85. $\int u^2\sqrt{a^2 + u^2}\,du$
$= \frac{u}{8}(a^2 + 2u^2)\sqrt{a^2 + u^2} - \frac{a^4}{8}\ln(u + \sqrt{a^2 + u^2}) + C$
86. $\int \frac{\sqrt{a^2 + u^2}}{u}\,du = \sqrt{a^2 + u^2} - a \ln\left|\frac{a + \sqrt{a^2 + u^2}}{u}\right| + C$
87. $\int \frac{\sqrt{a^2 + u^2}}{u^2}\,du = -\frac{\sqrt{a^2 + u^2}}{u} + \ln(u + \sqrt{a^2 + u^2}) + C$
88. $\int \frac{du}{\sqrt{a^2 + u^2}} = \ln(u + \sqrt{a^2 + u^2}) + C$
89. $\int \frac{u^2\,du}{\sqrt{a^2 + u^2}} = \frac{u}{2}\sqrt{a^2 + u^2} - \frac{a^2}{2}\ln(u + \sqrt{a^2 + u^2}) + C$
90. $\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2 + u^2} + a}{u}\right| + C$
91. $\int \frac{du}{u^2\sqrt{a^2 + u^2}} = -\frac{\sqrt{a^2 + u^2}}{a^2 u} + C$
92. $\int \frac{du}{(a^2 + u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2 + u^2}} + C$

Forms Involving $a + bu$

93. $\int \frac{u\,du}{a + bu} = \frac{1}{b^2}(a + bu - a \ln|a + bu|) + C$
94. $\int \frac{u^2\,du}{a + bu} = \frac{1}{2b^3}\left[(a + bu)^2 - 4a(a + bu) + 2a^2 \ln|a + bu|\right] + C$
95. $\int \frac{du}{u(a + bu)} = \frac{1}{a}\ln\left|\frac{u}{a + bu}\right| + C$
96. $\int \frac{du}{u^2(a + bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a + bu}{u}\right| + C$
97. $\int \frac{u\,du}{(a + bu)^2} = \frac{a}{b^2(a + bu)} + \frac{1}{b^2}\ln|a + bu| + C$
98. $\int \frac{du}{u(a + bu)^2} = \frac{1}{a(a + bu)} - \frac{1}{a^2}\ln\left|\frac{a + bu}{u}\right| + C$
99. $\int \frac{u^2\,du}{(a + bu)^2} = \frac{1}{b^3}\left(a + bu - \frac{a^2}{a + bu} - 2a \ln|a + bu|\right) + C$
100. $\int u\sqrt{a + bu}\,du = \frac{2}{15b^2}(3bu - 2a)(a + bu)^{3/2} + C$
101. $\int u^n\sqrt{a + bu}\,du$
$= \frac{2}{b(2n+3)}\left[u^n(a + bu)^{3/2} - na\int u^{n-1}\sqrt{a + bu}\,du\right]$
102. $\int \frac{u\,du}{\sqrt{a + bu}} = \frac{2}{3b^2}(bu - 2a)\sqrt{a + bu} + C$
103. $\int \frac{u^n\,du}{\sqrt{a + bu}} = \frac{2u^n\sqrt{a + bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a + bu}}$
104. $\int \frac{du}{u\sqrt{a + bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a + bu} - \sqrt{a}}{\sqrt{a + bu} + \sqrt{a}}\right| + C,\quad \text{if } a > 0$
$= \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a + bu}{-a}} + C,\quad \text{if } a < 0$
105. $\int \frac{du}{u^n\sqrt{a + bu}} = -\frac{\sqrt{a + bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a + bu}}$
106. $\int \frac{\sqrt{a + bu}}{u}\,du = 2\sqrt{a + bu} + a\int \frac{du}{u\sqrt{a + bu}}$
107. $\int \frac{\sqrt{a + bu}}{u^2}\,du = -\frac{\sqrt{a + bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a + bu}}$

Forms Involving $\sqrt{2au - u^2}$, $a > 0$

108. $\int \sqrt{2au - u^2}\,du = \frac{u - a}{2}\sqrt{2au - u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$
109. $\int u\sqrt{2au - u^2}\,du$
$= \frac{2u^2 - au - 3a^2}{6}\sqrt{2au - u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$
110. $\int \frac{du}{\sqrt{2au - u^2}} = \cos^{-1}\left(\frac{a - u}{a}\right) + C$
111. $\int \frac{du}{u\sqrt{2au - u^2}} = -\frac{\sqrt{2au - u^2}}{au} + C$

1. (**30 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   In the statements below, x, y, and z are the usual rectangular coordinates for $\mathbb{R}^3$, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in $\mathbb{R}^3$, $f = f(x, y)$ is a smoothly differentiable function, a, b, h, and k are numbers, $L = L(x, y)$ is the linearization of f at (a, b), t is a parameter, and $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a smooth curve.

   (a) $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.

   (b) Two lines in $\mathbb{R}^3$ define a plane.

   (c) If $f_{xx} = f_{yy} = 0$, then $f(a + h, b + k) = L(a + h, b + k)$.

   (d) If f has two local maxima, then it must have a local minimum.

   (e) The tangent plane to the graph of the function $z = f(x, y)$ at the point $(a, b, f(a, b))$ is the graph of the linearization of f at (a, b).

   (f) $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{b})(\mathbf{a} \bullet \mathbf{c})$.

   (g) $\displaystyle\int_{x=0}^{x=1} \int_{y=0}^{y=x} f\, dydx = \int_{y=0}^{y=1} \int_{x=0}^{x=1} f\, dxdy$.

   (h) $\mathbf{a}$ and $-2\mathbf{a}$ are parallel.

   (i) $\nabla f(a, b)$ is normal to the level curve of $f(x, y)$ passing through (a, b).

   (j) $\displaystyle\int_{t=0}^{t=1} \frac{d\mathbf{r}}{dt} dt = \mathbf{r}(1) - \mathbf{r}(0)$.

2. (**20 points**) Evaluate $\displaystyle\int_{x=0}^{x=9}\int_{y=0}^{y=4}(x+4y)^{\frac{1}{2}}\,dydx$.

3. (**20 points**) The integral $\displaystyle\int_{x=0}^{x=1}\int_{y=0}^{y=1}e^{xy}\,dydx$ cannot be evaluated algebraically.

   Approximate it using a Riemann sum with at least six summands. **Leave your answer as a Riemann sum**. Explain your work well enough that a reader could easily tell where everything is coming from.

4. (**40 points**) Consider the integral

$$I = \int_{x=1}^{x=3} \int_{y=1}^{y=x^2} xe^y \, dydx.$$

(a) Reverse the order of integration.

(b) Evaluate I, using whichever order seems most appropriate.

5. (**40 points**) Express **two** of the following three quantities as an iterated integral in an appropriate coordinate system. If you use coordinates other than rectangular, polar, cylindrical, or spherical, you must define them. **Do not evaluate**. Make it very clear which two integrals you want graded, for example by circling their letters or x-ing out the space (and any work) allotted to the third. I will not grade all three.

   (a) $\iiint\limits_T x + y - z \; dxdydz$, where T is the tetrahedron with vertices $P = (1, 1, 0)$, $Q = (1, 0, 1)$, $R = (0, 1, 1)$, and $S = (1, 1, 1)$. [You will probably find it useful to find the equations of the planes defined by three of these four points. You should be able to eyeball them; no justification is necessary.]

(b) The volume of the region in the first octant above the parallelogram P in the xy plane with sides $2x - y = 0$, $2y - x = 0$, $2x - y = 3$, and $2y - x = 3$, and below the hyperboloid $z = (2x - y)(2y - x)$. (You may assume the vertices of P are at $(0, 0)$, $(2, 1)$, $(3, 3)$, and $(1, 2)$ in counterclockwise order.)

(c) The mass of the region defined by the inequalities $x \geq 0$, $x^2 + y^2 \leq 9$, and $0 \leq z \leq 3y$, with density $25 - x^2 - y^2$.

6. (**Extra credit: 20 points**) In this problem, we will compute the area of the surface with equation $z^2 = x^2 + y^2 + 1$ above the rectangle R defined by $1 \leq x \leq 2$, $2 \leq y \leq 3$.

   (a) Use the chain rule to find the relationship between dx, dy, and dz on this surface.

   (b) If $(x, y, z = \sqrt{x^2 + y^2 + 1})$ is a point on this surface, find the nearby z-coordinates z_1, z_2, and z_3 such that $(x + dx, y, z_1)$, $(x + dx, y + dy, z_2)$, and $(x, y + dy, z_3)$ are also on the surface. [Hint: In every case the answer is "$z + dz$". But that won't earn you any credit. Rephrase it using x, y, z, dx, and dy.]

   (c) Find the area of the parallelogram with vertices (x, y, z), $(x+dx, y, z_1)$, $(x + dx, y + dy, z_2)$, and $(x, y + dy, z_3)$. [Hint: this is probably most easily done using techniques from chapter 12.]

   (d) Express the sum of infinitely many such areas as an integral. **Do not evaluate.**