 Name:____________________

Math 2163

Jeff Mermin's section, Test 1, September 16

On the essay questions (# 2–7) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

Do not evaluate any integrals on this test. If you would take an integral, instead simplify the integrand and the limits of integration (if any), and leave the integral as your final answer.

1. (**30 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   In the statements below, a, b, c, p, q, and r are numbers, x, y, and z are coordinates for $\mathbb{R}^3$, t is a parameter, P is a point in $\mathbb{R}^3$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^3$, $\mathbf{r} = \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ and $\mathbf{s} = \mathbf{s}(t)$ are curves in $\mathbb{R}^3$,

   (a) The line with equation $(x, y, z) = (p, q, r) + \langle a, b, c\rangle$ is contained in the plane with equation $a(x-p) + b(y-q) + c(z-r) = d$.

   (b) $\mathbf{v}$ and $\mathbf{w}$ are parallel if and only if $\mathbf{v} \bullet \mathbf{w} = \mathbf{0}$.

   (c) If ℓ is the line through $P = (1, 2, 3)$ and $Q = (4, 5, 6)$, then $\langle 4, 5, 6\rangle$ is a direction vector for ℓ.

   (d) $|\mathbf{v} \bullet \mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|$.

   (e) $\dfrac{d}{dt}(\mathbf{r} \bullet \mathbf{s}) = \dfrac{d\mathbf{r}}{dt} \bullet \dfrac{d\mathbf{s}}{dt}$.

   (f) $\dfrac{d\mathbf{r}}{dt} = \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt} \right\rangle$.

   (g) $\|\mathbf{v}\| + \|\mathbf{w}\| = \|\mathbf{v} + \mathbf{w}\|$.

   (h) The equations $(x, y, z) = P + t\mathbf{v}$ and $(x, y, z) = P - t\mathbf{v}$ define the same line.

   (i) The equations $x = 2$, $y = -1$, $z = 0$ define a line in $\mathbb{R}^3$.

   (j) If two distinct planes in $\mathbb{R}^3$ intersect, they intersect in a line.

2. (**20 points**) Let $\mathbf{x} = \langle 2, 5, 5 \rangle$ and $\mathbf{y} = \langle 4, 5, 2 \rangle$. Compute the following:

   (a) $-3\mathbf{x} - 3\mathbf{y}$.

   (b) $\mathbf{x} \bullet \mathbf{y}$.

   (c) $\mathbf{x} \times \mathbf{y}$.

   (d) $(-3\mathbf{x} - 3\mathbf{y}) \times \mathbf{y}$.

3. (**30 points**) Let $P = (-3, -3, -2)$, $Q = (4, -3, 1)$, and $R = (-1, 1, 2)$.

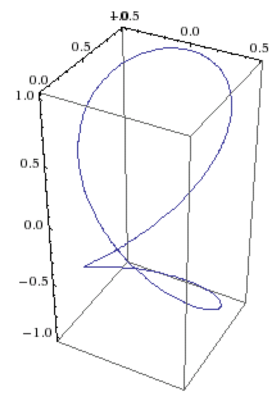
(a) Find a parametrization of the line through P and Q.

(b) Find an equation for the plane containing P, Q, and R.

4. (**20 points**) Find the distance from the line $\ell : (x, y, z) = (2, -2, 1) + (-1, 4, -1)t$ to the point $S = (-3, -1, 1)$. Explain the geometric significance of any points, vectors, lines, or planes that you find along the way.

5. (**20 points**) Find an equation for the plane consisting of all points equidistant from $T = (4, 1, 5)$ and $U = (-3, -2, -1)$. Explain the geometric significance of any points, vectors, lines, or planes that you find along the way.

6. (**30 points**) Consider Viviani's curve, which has parametric equation $(x, y, z) = \left(\dfrac{\sin 2t}{2}, \sin^2 t, \sin t\right)$. Wolfram Alpha's graph is shown below.



(a) Find a parametrization of the tangent line to Viviani's curve at the point $\left(\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{\sqrt{2}}\right)$.

(b) Find the length of Viviani's curve.

7. (**Extra credit: 20 points**) If P is a polygon in $\mathbb{R}^2$ with vertices (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) in counterclockwise order, we can compute its area by the following procedure:

   First, create an $(n+1) \times 2$ matrix $M = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{bmatrix}$ by stacking the coordinates on top of each other (in order), and repeating the first point at the bottom. Then set D equal to the "sum of the downward diagonals", $D = x_1y_2 + x_2y_3 + \cdots + x_{n-1}y_n + x_ny_1$, and set U equal to the "sum of the upward diagonals" in a similar way.

   Then the area is given by $A = \dfrac{D-U}{2}$.

   (a) Verify that this procedure is correct by using it to compute the area of the triangle with vertices $(-1, -3)$, $(2, 3)$, and $(2, 5)$, then computing the area in some other way.

(b) Use cross products to prove the procedure is correct for a triangle with vertices (a, b), (c, d), and (e, f). (Remember, cross products are only defined for vectors in $\mathbb{R}^3$, so you're going to have to do something clever.

(c) Prove the procedure is correct for quadrilaterals. (You may assume it works for triangles, even if you haven't done part (b).)