Math 2163

Jeff Mermin's section, Final exam, December 8

On the essay questions (# 2–14) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

You probably won't find the integral tables on the following pages helpful, but here they are anyway.

Basic Forms

1. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
2. $\int \frac{du}{u} = \ln|u| + C$
3. $\int e^u\,du = e^u + C$
4. $\int a^u\,du = \frac{a^u}{\ln a} + C$
5. $\int \sin u\,du = -\cos u + C$
6. $\int \cos u\,du = \sin u + C$
7. $\int \sec^2 u\,du = \tan u + C$
8. $\int \csc^2 u\,du = -\cot u + C$
9. $\int \sec u \tan u\,du = \sec u + C$
10. $\int \csc u \cot u\,du = -\csc u + C$
11. $\int \tan u\,du = \ln|\sec u| + C$
12. $\int \cot u\,du = \ln|\sin u| + C$
13. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
14. $\int \csc u\,du = \ln|\csc u - \cot u| + C$
15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$
16. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

Exponential and Logarithmic Forms

17. $\int ue^{au}\,du = \frac{1}{a^2}(au - 1)e^{au} + C$
18. $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$
19. $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2 + b^2}(a \sin bu - b \cos bu) + C$
20. $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2 + b^2}(a \cos bu + b \sin bu) + C$
21. $\int \ln u\,du = u \ln u - u + C$
22. $\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$
23. $\int \frac{1}{u \ln u}\,du = \ln|\ln u| + C$

Hyperbolic Forms

24. $\int \sinh u\,du = \cosh u + C$
25. $\int \cosh u\,du = \sinh u + C$
26. $\int \tanh u\,du = \ln \cosh u + C$
27. $\int \coth u\,du = \ln|\sinh u| + C$
28. $\int \operatorname{sech} u\,du = \tan^{-1}|\sinh u| + C$
29. $\int \operatorname{csch} u\,du = \ln\left|\tanh \frac{1}{2}u\right| + C$
30. $\int \operatorname{sech}^2 u\,du = \tanh u + C$
31. $\int \operatorname{csch}^2 u\,du = -\coth u + C$
32. $\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$
33. $\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$

Trigonometric Forms

34. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$
35. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$
36. $\int \tan^2 u\,du = \tan u - u + C$
37. $\int \cot^2 u\,du = -\cot u - u + C$
38. $\int \sin^3 u\,du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$
39. $\int \cos^3 u\,du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$
40. $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$
41. $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$
42. $\int \sec^3 u\,du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

$\int \csc^3 u\,du = -\frac{1}{n}\csc u \cot u + \frac{1}{n}\ln|\csc u - \cot u| + C$

$\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$

$\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$

$\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$

$\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\,du$

$\int \sec^n u\,du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$

$\int \csc^n u\,du = \frac{-1}{n-1}\cot u \csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\,du$

$\int \sin au \sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

$\int \cos au \cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

$\int \sin au \cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

$\int u \sin u\,du = \sin u - u \cos u + C$

$\int u \cos u\,du = \cos u + u \sin u + C$

$\int u^n \sin u\,du = -u^n \cos u + n\int u^{n-1}\cos u\,du$

$\int u^n \cos u\,du = u^n \sin u - n\int u^{n-1}\sin u\,du$

$\int \sin^n u \cos^m u\,du$

$= -\frac{\sin^{n-1} u \cos^{m+1} u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} u \cos^m u\,du$

$= \frac{\sin^{n+1} u \cos^{m-1} u}{n+m} + \frac{m-1}{n+m}\int \sin^n u \cos^{m-2} u\,du$

rse Trigonometric Forms

$$\int \sin^{-1} u\, du = u \sin^{-1} u + \sqrt{1-u^2} + C$$

$$\int \cos^{-1} u\, du = u \cos^{-1} u - \sqrt{1-u^2} + C$$

$$\int \tan^{-1} u\, du = u \tan^{-1} u - \frac{1}{2}\ln(1+u^2) + C$$

$$\int u \sin^{-1} u\, du = \frac{2u^2-1}{4}\sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$$

$$\int u \cos^{-1} u\, du = \frac{2u^2-1}{4}\cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$$

$$\int u \tan^{-1} u\, du = \frac{u^2+1}{2}\tan^{-1} u - \frac{u}{2} + C$$

$$\int u^n \sin^{-1} u\, du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1} u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$$

$$\int u^n \cos^{-1} u\, du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1} u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$$

$$\int u^n \tan^{-1} u\, du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1} u - \int \frac{u^{n+1}\,du}{1+u^2}\right], \quad n \neq -1$$

Forms Involving $\sqrt{a^2-u^2}$, $a > 0$

67. $\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

68. $\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

69. $\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

70. $\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2-u^2} - \sin^{-1}\frac{u}{a} + C$

71. $\int \frac{u^2\,du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

72. $\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

73. $\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{1}{a^2u}\sqrt{a^2-u^2} + C$

74. $\int (a^2-u^2)^{3/2}\,du = -\frac{u}{8}(2u^2-5a^2)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

75. $\int \frac{du}{(a^2-u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2-u^2}} + C$

Forms Involving $\sqrt{u^2-a^2}$, $a > 0$

76. $\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

77. $\int u^2\sqrt{u^2-a^2}\,du$
$= \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

78. $\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\cos^{-1}\frac{a}{|u|} + C$

79. $\int \frac{\sqrt{u^2-a^2}}{u}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$

80. $\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$

81. $\int \frac{u^2\,du}{\sqrt{u^2-a^2}} = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

82. $\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C$

83. $\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$

Forms Involving $\sqrt{a^2+u^2}$, $a > 0$

84. $\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

85. $\int u^2\sqrt{a^2+u^2}\,du$
$= \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

86. $\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

87. $\int \frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u+\sqrt{a^2+u^2}\right) + C$

88. $\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u+\sqrt{a^2+u^2}\right) + C$

89. $\int \frac{u^2\,du}{\sqrt{a^2+u^2}} = \frac{u}{2}\sqrt{a^2+u^2} - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

90. $\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

91. $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$

92. $\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

Forms Involving $a + bu$

93. $\int \frac{u\,du}{a+bu} = \frac{1}{b^2}\left(a+bu-a\ln|a+bu|\right) + C$

94. $\int \frac{u^2\,du}{a+bu} = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$

95. $\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

96. $\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

97. $\int \frac{u\,du}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$

98. $\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

99. $\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left(a+bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$

100. $\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2} + C$

101. $\int u^n\sqrt{a+bu}\,du$
$= \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$

102. $\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$

103. $\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a+bu}}$

104. $\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, \quad \text{if } a > 0$
$= \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}} + C, \quad \text{if } a < 0$

105. $\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$

106. $\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$

107. $\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$

Forms Involving $\sqrt{2au-u^2}$, $a > 0$

108. $\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

109. $\int u\sqrt{2au-u^2}\,du$
$= \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

110. $\int \frac{du}{\sqrt{2au-u^2}} = \cos^{-1}\left(\frac{a-u}{a}\right) + C$

111. $\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C$

1. (**50 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   In the statements below, x, y, z are the usual rectangular coordinates, u and v are alternative coordinates for the xy plane, t is a parameter, $\mathbf{v}$ and $\mathbf{w}$ are vectors, $\mathbf{r}(t) = \langle x_{\mathbf{r}}(t), y_{\mathbf{r}}(t), z_{\mathbf{r}}(t)\rangle$ and $\mathbf{s}(t) = \langle x_{\mathbf{s}}(t), y_{\mathbf{s}}(t), z_{\mathbf{s}}(t)\rangle$ are parametrized curves, and f is a smoothly differentiable function.

   (a) Two lines in $\mathbb{R}^3$ define a plane.

   (b) If $R = \{(x, y, z) : 1 \le x^2 + y^2 + z^2 \le 16\}$, then $\iiint\limits_R dV = 84\pi$.

   (c) $\dfrac{d}{dt}(\mathbf{r} \times \mathbf{s}) = \dfrac{d\mathbf{r}}{dt} \times \dfrac{d\mathbf{s}}{dt}$.

   (d) $\mathbf{v}$ and $\mathbf{w}$ are perpendicular if and only if $\mathbf{v} \times \mathbf{w} = \mathbf{0}$.

   (e) $\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 1.$

(f) There are functions $h_1(x)$ and $h_2(y)$ such that $f(x,y) = h_1(x) + h_2(y)$.

(g) If $z = f(x,y)$, then $z_t = z_x x_t + z_y y_t$.

(h) There is a function $g(x,y)$ such that $g_x(x,y) = x^2 + y^2$ and $g_y(x,y) = x^2 - y^2$.

(i) $|\mathbf{v} \bullet \mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|$.

(j) If $F_1(x,y)$ and $F_2(x,y)$ are both antiderivatives of $f(x,y)$ with respect to x (that is, $\frac{d}{dx}(F_1) = \frac{d}{dx}(F_2) = f$), then $F_1(x,y) - F_2(x,y)$ is a constant function.

2. (**20 points**) Let $\mathbf{x} = \langle 4, -2, 5 \rangle$ and $\mathbf{y} = \langle 1, 3, 3 \rangle$.

   (a) Compute $\mathbf{x} \bullet \mathbf{y}$.

   (b) Compute $\mathbf{x} \times \mathbf{y}$.

3. (**10 points**) Find a point on the intersection of the two planes $4x+4y-z = 2$ and $2x + y + z = 3$.

4. (**10 points**) Find an equation for the plane consisting of all points equidistant from $P = (2, 4, 5)$ and $Q = (0, -1, -3)$.

5. (**20 points**) Find an equation for the tangent plane to the surface $z = e^{-3y} \sin x$ at the point $(\pi, 0, 0)$.

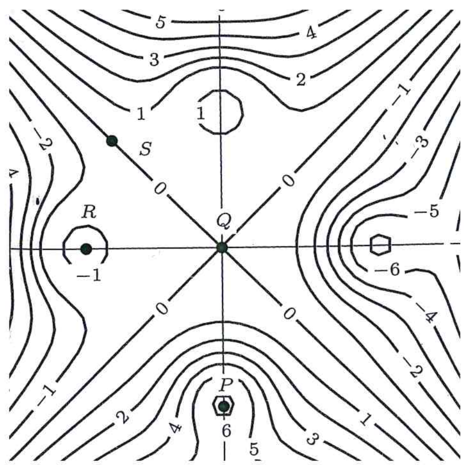
6. (**40 points**) The **spiral of Archimedes** is the curve with polar equation $r = \theta$.

   (a) Verify that $\langle x, y \rangle = \langle t\cos t, t\sin t \rangle$ is a correct parametrization for the spiral of Archimedes.

   (b) Find the length of the first turn of the spiral, from the point P with rectangular coordinates $(0, 0)$ to the point Q with rectangular coordinates $(2\pi, 0)$. Express your answer as a definite integral and **do not evaluate**.

   (c) Find a parametrization of the tangent line to the spiral of Archimedes at Q.

7. (**20 points**) Consider the function $z = f(x, y)$ whose contour plot is shown below.



Determine whether each of the marked points is a critical point or not. If it is, is it a local maximum, local minimum, or saddle point? (**No justification is necessary on this problem**, but incorrect answers with sensible justification may earn partial credit.)

(a) $P \approx (0, -1)$.

(b) $Q \approx (0, 0)$.

(c) $R \approx (-1, 0)$.

(d) $S \approx (-1, 1)$.

8. (**20 points**) Evaluate $\displaystyle\int_{y=0}^{y=1} \int_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} y\, dx dy$.

9. (**20 points**) Express the volume bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 2$ as an iterated integral. **Do not evaluate**.

10. **(10 points)** Prove that the vector field $F(x,y) = \langle x^3, y^2 \rangle$ is conservative. Then find a potential function $f(x,y)$ satisfying $F = \nabla f$.

11. **(20 points)** Let C be the path starting at the origin, going in a straight line to $(1,0)$, then in another straight line to $(0,1)$, then in a straight line back to the origin. Compute the following integrals.

    (You can do these without integrating anything if you use the right theorems - but be sure to say which theorems you're using and how. Alternatively, you can do them directly, by the usual parametrize-and-integrate method.)

    (a) $\int_C \langle x^3, y^2 \rangle \bullet d\mathbf{s}$

    (b) $\int_C x\,dy$

12. (**10 points**) Sketch regions in the plane that are:

    (a) Simply connected.

    (b) Connected but not simply connected.

    (c) Not connected.

13. (**Extra credit: 10 points**) In the space remaining on this page, write the largest integer you can. You do not have to simplify, as, for example, "10^3" may take up less space than "1000". However, I should be able (at least in theory) to determine the precise value of your number with no ambiguity or reference to the experiential universe (so, for example, "the number of stars in the sky" is right out), so you may need to use some space defining your notation.

14. (**Extra credit: 10 points**) Let $A = (a, \frac{1}{a})$ and $B = (b, \frac{1}{b})$ be points on the curve $xy = 1$ in the first quadrant. Let $C = (c, \frac{1}{c})$ be another point between A and B on the same curve.

   (a) Find the area of triangle ABC. (We have seen at least four ways to compute this area in class. Your answer should be an expression in a, b, and c that an eighth-grader would understand.)

   (b) Find the value of c (in terms of a and b) that maximizes the area found above.