Math 2163

Jeff Mermin's section, Test 2, October 15

On the essay questions (# 3–9) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

Do not evaluate any integrals on this test. If you would take an integral, instead simplify the integrand and the limits of integration (if any), and leave the integral as your final answer.

1. (**15 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   In the statements below, a, b, h, and k are numbers, x, y, and z are variables, dx, dy, and dz represent differentials, $f = f(x, y)$ and $g = g(x, y)$ are smooth functions, $L = L(x, y)$ is the linearization of f at (a, b), and $\mathbf{x}$ and $\mathbf{y}$ are vectors.

   (a) The vector $\langle dx, dy, dz \rangle$ is normal to the graph of $z = f(x, y)$.

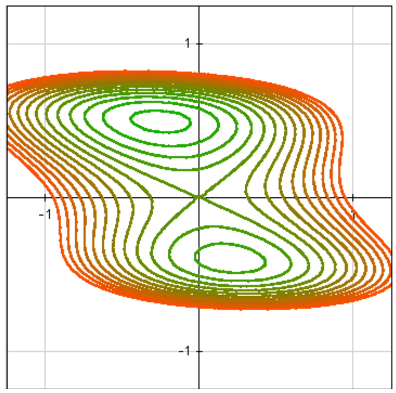
   (b) $\nabla(f + g) = \nabla f + \nabla g$.

   (c) $\mathbf{x} - \mathbf{y} = \mathbf{y} - \mathbf{x}$.

   (d) Two lines in $\mathbb{R}^3$ define a plane.

   (e) If $f_{xx} = f_{yy} = 0$, then $f(a + h, b + k) = L(a + h, b + k)$.

2. (**25 points**) Consider the function $z = f(x, y)$ whose contour map is shown below.



No justification is necessary for correct answers to the questions below. However, incorrect answers with some good justification may earn partial credit.

(a) Assume this function has no local maxima. Are there any local minima and/or saddle points? If so, identify and label them on the graph.

(b) Draw in a vector representing ∇f at the point $\left(-1, \dfrac{1}{3}\right)$.

(c) Which is larger: $f_y\left(0, \dfrac{3}{4}\right)$ or $f_y\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$?

3. (**30 points**) Find all the second partial derivatives of the function

$$f(x, y) = \ln\left(\frac{x^2 - y^2}{x}\right)$$

4. (**20 points**) Find an equation for the tangent plane to the graph of the function $z = xy + 2x^2y + x^3y^2$ at the point $(1, 2, 10)$.

5. (**20 points**) Use a linear approximation technique from this class to approximate the value of $\dfrac{499.975}{10.0008}$.

   (Maybe you know other techniques for efficient division. If so, humor me and use one from this class anyway: I could have asked a question that didn't admit your techniques, but it would be harder.)

6. (**10 points**) Using any appropriate method, find $\dfrac{dz}{dt}$, if

$$z = \sin\left(\frac{(\cos t + \sin t)^3}{\left(\frac{t^2-4t}{t^2}\right)}\right) + (\cos t + \sin t)^2 - 3\,(\cos t + \sin t)\left(\frac{t^2-4t}{t^2}\right)$$

7. (**15 points**) Let $z = e^{xy-y^2}$, $P = (3,3)$, and $\mathbf{v} = \langle 2, -1\rangle$. Find the directional derivative $D_{\mathbf{u}}(z)(P)$, if $\mathbf{u}$ is the unit vector in the direction of $\mathbf{v}$.

8. (**15 points**)

Let $f(x,y) = \sin(x+y) - \cos(x)$ You do not need to compute the partial derivatives:

$$f_x = \cos(x+y) + \sin(x) f_y = \cos(x+y)$$

Determine, with justification, which, if any, of the points below are critical points of f.

(a) $P = \left(\frac{7\pi}{4}, \frac{3\pi}{4}\right)$

(b) $Q = \left(\pi, \frac{-\pi}{2}\right)$

(c) $R = \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

9. (**Extra credit: 20 points**) Recall that the **degree n Taylor Polynomial** (centered at 0) for a function $f(x)$ is $T_n(x) = \sum_{d=0}^{d=n} \frac{f^{(d)}(0)}{d!} x^d$.

Also recall that, if f is a polynomial of degree at most n, the degree n Taylor polynomial of f is just f itself.

(a) What should be the degree three Taylor polynomial (centered at the origin) for the function $f(x, y) = x^3 + x^2y + xy$?

(b) Propose a sensible formula for the degree n Taylor polynomial (centered at the origin) for an arbitrary (smoothly differentiable) function $f(x, y)$.