Math 2163

Jeff Mermin's section, Test 1, September 17

On the essay questions (# 2–10) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

Do not evaluate any integrals on this test. If you would take an integral, instead simplify the integrand and the limits of integration (if any), and leave the integral as your final answer.

1. (**15 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   In the statements below, a is a real number, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^3$, t is a parameter, x, y, z and x', y', z' are variables which are each differentiable functions of t, and $\mathbf{r} = \mathbf{r}(t) = (x(t), y(t), z(t))$ and $\mathbf{s} = \mathbf{s}(t) = (x'(t), y'(t), z'(t))$ are parametrized curves.

   (a) $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$.

   (b) $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\mathbf{u} \times \mathbf{v} = 0$.

   (c) $\displaystyle\int_{t=0}^{t=1} \frac{d\mathbf{r}}{dt} dt = \mathbf{r}(1) - \mathbf{r}(0)$.

   (d) $\frac{d}{dt}(\mathbf{r} \times \mathbf{s}) = \frac{d}{dt}\mathbf{r} \times \frac{d}{dt}\mathbf{s}$.

   (e) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

2. (**20 points**) Let $\mathbf{x} = \langle 6, -4, 4 \rangle$ and $\mathbf{y} = \langle -1, -2, -3 \rangle$. Compute the following:

   (a) $2\mathbf{x} + 3\mathbf{y}$.

   (b) $\mathbf{x} \bullet \mathbf{y}$.

   (c) $\mathbf{x} \times \mathbf{y}$.

   (d) $(4\mathbf{x} - 2\mathbf{y}) \times \mathbf{x}$.

3. (**10 points**) Find two points on the plane $F : 8x + 7y + 9z = 1$

4. (**20 points**) Consider the line $\ell : (x, y, z) = (5, 1, 0) + (2, -2, -4)t$ and the plane $G : 9x + 3y + 3z = 6$. Determine whether or not they intersect. If they do, find the point of intersection. If they do not, find the distance from ℓ to G.

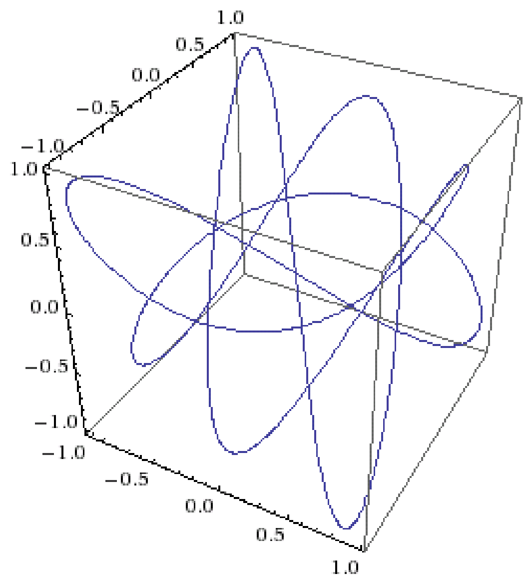
5. (**10 points**) Find a parametrization of the line passing through $P = (1, -5, -5)$ and $Q = (-3, -1, 6)$.

6. (**20 points**) Find an equation of the plane containing the parallel lines $m_1 : (x, y, z) = (2, -5, 5) + (-3, 1, 9)t$ and $m_2 : (x, y, z) = (0, -5, 7) + (-3, 1, 9)t$.

7. (**20 points**) Consider the point $R = (0, -2, 1)$ and the plane $H : y - x = 3$. Find the point S on H which is as close as possible to R.

8. (**20 points**) Find a parametrization of the tangent line to $\mathbf{r}(t) = \left(2t, t^2, \frac{1}{3}t^3\right)$ at the point $P = (6, 9, 9)$.

9. (**15 points**) The parametric curve $\mathbf{r}(t) = (\cos 3t, \sin 4t, \cos 5t)$, graphed below by Wolfram Alpha, forms a closed loop with period 2π. Find its length.



10. (**Extra credit: 20 points**) Prove the following famous inequalities from arithmetic. You may use algebra, geometry, calculus, or the other inequalities, but you can't use one of the inequalities to prove itself.

    (a) (**The triangle inequality**) If **x** and **y** are vectors, then

    $$|x + y| \leq |x| + |y|.$$

    (b) (**The Cauchy-Schwarz inequality**) If $(x_1, x_2, \ldots, x_n)$ and $y_1, y_2, \ldots, y_n)$ are two lists of n numbers, then

    $$(x_1y_1 + x_2y_2 + \cdots + x_ny_n)^2 \leq (x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2).$$

    (c) (**The RMS inequality**) If a and b are numbers, then

    $$\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}.$$

    (d) (**The GM inequality**) If a and b are positive numbers, then

    $$\sqrt{ab} \leq \frac{a+b}{2}.$$