# Math 2163

Jeff Mermin's section, Test 3, November 22

On the essay questions (# 2–6) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

You may or may not find the following tables helpful.

### Basic Forms

1. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
2. $\int \frac{du}{u} = \ln|u| + C$
3. $\int e^u\,du = e^u + C$
4. $\int a^u\,du = \frac{a^u}{\ln a} + C$
5. $\int \sin u\,du = -\cos u + C$
6. $\int \cos u\,du = \sin u + C$
7. $\int \sec^2 u\,du = \tan u + C$
8. $\int \csc^2 u\,du = -\cot u + C$
9. $\int \sec u \tan u\,du = \sec u + C$
10. $\int \csc u \cot u\,du = -\csc u + C$
11. $\int \tan u\,du = \ln|\sec u| + C$
12. $\int \cot u\,du = \ln|\sin u| + C$
13. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
14. $\int \csc u\,du = \ln|\csc u - \cot u| + C$
15. $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$
16. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

### Exponential and Logarithmic Forms

17. $\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au} + C$
18. $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$
19. $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$
20. $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$
21. $\int \ln u\,du = u\ln u - u + C$
22. $\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$
23. $\int \frac{1}{u\ln u}\,du = \ln|\ln u| + C$

### Hyperbolic Forms

24. $\int \sinh u\,du = \cosh u + C$
25. $\int \cosh u\,du = \sinh u + C$
26. $\int \tanh u\,du = \ln\cosh u + C$
27. $\int \coth u\,du = \ln|\sinh u| + C$
28. $\int \operatorname{sech} u\,du = \tan^{-1}|\sinh u| + C$
29. $\int \operatorname{csch} u\,du = \ln\left|\tanh\frac{1}{2}u\right| + C$
30. $\int \operatorname{sech}^2 u\,du = \tanh u + C$
31. $\int \operatorname{csch}^2 u\,du = -\coth u + C$
32. $\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$
33. $\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$

### Trigonometric Forms

34. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$
35. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$
36. $\int \tan^2 u\,du = \tan u - u + C$
37. $\int \cot^2 u\,du = -\cot u - u + C$
38. $\int \sin^3 u\,du = -\frac{1}{3}(2+\sin^2 u)\cos u + C$
39. $\int \cos^3 u\,du = \frac{1}{3}(2+\cos^2 u)\sin u + C$
40. $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$
41. $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$
42. $\int \sec^3 u\,du = \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$
43. $\int \csc^3 u\,du = -\frac{1}{n}\csc u\cot u + \frac{1}{n}\ln|\csc u - \cot u| + C$
44. $\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1}u\cos u + \frac{n-1}{n}\int \sin^{n-2}u\,du$
45. $\int \cos^n u\,du = \frac{1}{n}\cos^{n-1}u\sin u + \frac{n-1}{n}\int \cos^{n-2}u\,du$
46. $\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1}u - \int \tan^{n-2}u\,du$
47. $\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1}u - \int \cot^{n-2}u\,du$
48. $\int \sec^n u\,du = \frac{1}{n-1}\tan u\sec^{n-2}u + \frac{n-2}{n-1}\int \sec^{n-2}u\,du$
49. $\int \csc^n u\,du = \frac{-1}{n-1}\cot u\csc^{n-2}u + \frac{n-2}{n-1}\int \csc^{n-2}u\,du$
50. $\int \sin au\sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$
51. $\int \cos au\cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$
52. $\int \sin au\cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$
53. $\int u\sin u\,du = \sin u - u\cos u + C$
54. $\int u\cos u\,du = \cos u + u\sin u + C$
55. $\int u^n\sin u\,du = -u^n\cos u + n\int u^{n-1}\cos u\,du$
56. $\int u^n\cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$
57. $\int \sin^n u\cos^m u\,du$
$= -\frac{\sin^{n-1}u\cos^{m+1}u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2}u\cos^m u\,du$
$= \frac{\sin^{n+1}u\cos^{m-1}u}{n+m} + \frac{m-1}{n+m}\int \sin^n u\cos^{m-2}u\,du$

### Inverse Trigonometric Forms

58. $\int \sin^{-1}u\,du = u\sin^{-1}u + \sqrt{1-u^2} + C$
59. $\int \cos^{-1}u\,du = u\cos^{-1}u - \sqrt{1-u^2} + C$
60. $\int \tan^{-1}u\,du = u\tan^{-1}u - \frac{1}{2}\ln(1+u^2) + C$
61. $\int u\sin^{-1}u\,du = \frac{2u^2-1}{4}\sin^{-1}u + \frac{u\sqrt{1-u^2}}{4} + C$
62. $\int u\cos^{-1}u\,du = \frac{2u^2-1}{4}\cos^{-1}u - \frac{u\sqrt{1-u^2}}{4} + C$
63. $\int u\tan^{-1}u\,du = \frac{u^2+1}{2}\tan^{-1}u - \frac{u}{2} + C$
64. $\int u^n\sin^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1}u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$
65. $\int u^n\cos^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1}u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$
66. $\int u^n\tan^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1}u - \int \frac{u^{n+1}\,du}{1+u^2}\right], \quad n \neq -1$

### Forms Involving $\sqrt{a^2-u^2}$, $a > 0$

67. $\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$
68. $\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$
69. $\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$
70. $\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2-u^2} - \sin^{-1}\frac{u}{a} + C$
71. $\int \frac{u^2\,du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$
72. $\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$
73. $\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{1}{a^2u}\sqrt{a^2-u^2} + C$
74. $\int (a^2-u^2)^{3/2}\,du = -\frac{u}{8}(2u^2-5a^2)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} +$
75. $\int \frac{du}{(a^2-u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2-u^2}} + C$

### Forms Involving $\sqrt{u^2-a^2}$, $a > 0$

76. $\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$
77. $\int u^2\sqrt{u^2-a^2}\,du$
$= \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$
78. $\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\cos^{-1}\frac{a}{|u|} + C$
79. $\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$
80. $\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$
81. $\int \frac{u^2\,du}{\sqrt{u^2-a^2}} = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$
82. $\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C$
83. $\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$

### Forms Involving $\sqrt{a^2+u^2}$, $a > 0$

84. $\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$
85. $\int u^2\sqrt{a^2+u^2}\,du$
$= \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right) + C$
86. $\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$
87. $\int \frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u+\sqrt{a^2+u^2}\right) + C$
88. $\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u+\sqrt{a^2+u^2}\right) + C$
89. $\int \frac{u^2\,du}{\sqrt{a^2+u^2}} = \frac{u}{2}\sqrt{a^2+u^2} - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$
90. $\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$
91. $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$
92. $\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

### Forms Involving $a + bu$

93. $\int \frac{u\,du}{a+bu} = \frac{1}{b^2}\left(a+bu-a\ln|a+bu|\right) + C$
94. $\int \frac{u^2\,du}{a+bu} = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$
95. $\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$
96. $\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$
97. $\int \frac{u\,du}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$
98. $\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$
99. $\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left(a+bu-\frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$
100. $\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2} + C$
101. $\int u^n\sqrt{a+bu}\,du$
$= \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$
102. $\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$
103. $\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a+bu}}$
104. $\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, \quad \text{if } a > 0$
$= \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}} + C, \quad \text{if } a < 0$
105. $\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$
106. $\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$
107. $\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$

### Forms Involving $\sqrt{2au-u^2}$, $a > 0$

108. $\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$
109. $\int u\sqrt{2au-u^2}\,du$
$= \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$
110. $\int \frac{du}{\sqrt{2au-u^2}} = \cos^{-1}\left(\frac{a-u}{a}\right) + C$
111. $\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C$

1. (**30 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   In the statements below, $a$, $b$, $c$, $d$, and $d'$ are numbers, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors, $t$ is a variable, $x$, $y$, and $z$ are rectangular coordinates for $\mathbb{R}^3$, $f$ is a smooth function defined everywhere, and $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a smooth curve in $\mathbb{R}^3$. $D$ is a region in $\mathbb{R}^2$ or $\mathbb{R}^3$. $dV$ means what it does in the book.

   **(a)** The vector $\langle dx, dy, dz\rangle$ is normal to the graph of $z = f(x, y)$.

   **(b)** If $w = h(z)$ is a differentiable function of $z$, and $z = f(x, y)$, then
   $$\frac{\partial w}{\partial x} = \frac{dw}{dz}\frac{\partial z}{\partial x}.$$

   **(c)** The distance between the planes $F : ax + by + cz = d$ and $G : ax + by + cz = d'$ is $|d - d'|$.

   **(d)** If $D$ is closed and bounded, then $f$ has a global maximum on $D$.

   **(e)** $\mathbf{v}$ and $-2\mathbf{v}$ point in the same direction.

   **(f)** If $D = \{(x, y, z) : x^2 + y^2 + z^2 \leq 9\}$, then $\iiint_D dV = 36\pi$.

   **(g)** $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{b})(\mathbf{a} \bullet \mathbf{c})$.

   **(h)** $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.

   **(i)** $\mathbf{v}$ and $\mathbf{w}$ are perpendicular if and only if $\mathbf{v} \times \mathbf{w} = \mathbf{0}$.

   **(j)** $\frac{d\mathbf{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$.

2. (**30 points**) Evaluate the integrals.

**(a)** $\displaystyle\int_{x=-1}^{x=1}\int_{y=0}^{y=2} 4+x^2-y^2\,dydx$

**(b)** $\displaystyle\int_{y=0}^{y=1}\int_{x=y^3}^{x=y^2} 2x+y^2\,dxdy$

3. (**15 points**) Consider the integral $I = \int_{y=0}^{y=8} \int_{x=\sqrt[3]{y}}^{x=2} e^{x^4}\, dx dy$.

**(a)** Reverse the order of integration.

**(b)** Evaluate $I$, using whichever order seems most appropriate.

4. (**15 points**) Some values of a continuous function $f(x, y)$ on the rectangle $R = \{0 \le x \le 30, 0 \le y \le 20\}$ are given in the table below. (Apparently $f$ is hard to compute, because some values are unknown). Estimate the value of $\iint_R f(x, y)dA$ using a Riemann sum with at least six summands.

   **Leave your answer as a Riemann sum**.

| $y$ \ $x$ | 0 | 5 | 10 | 15 | 20 | 25 | 30 |
|---|---|---|---|---|---|---|---|
| 0 | 2 | ? | 4 | ? | ? | ? | 8 |
| 5 | 2 | ? | 4 | ? | 8 | 10 | ? |
| 10 | ? | ? | ? | 8 | ? | ? | ? |
| 15 | 2 | 3 | ? | ? | 6 | 8 | 7 |
| 20 | 2 | ? | 2 | 2 | ? | ? | 4 |

5. (**60 points**) Express each of the following as an iterated integral in an appropriate coordinate system. If you use coordinates other than rectangular, polar, cylindrical, or spherical, you must define them. **Do not evaluate**.

   **(a)** The volume of the region defined by the inequalities

$$0 \leq z \leq 18 - 2x^2 - 2y^2.$$

   **(b)** $\iiint\limits_R xe^{yz}\,dx\,dy\,dz$, where $R$ is the region defined by the inequalities $\{x \geq 0, y \geq 0, z \geq 0, x + y \leq 1, x^2 + z \leq 1\}$.

**(c)** $\iint_R xy\,dydx$, where $R$ is the region defined by the inequalities

$$1 \le xy \le 3 \qquad \text{and} \qquad x \le y \le 3x.$$

6. (**Extra credit: 20 points**) Earth is facing a serious werewolf crisis, and in a last-ditch attempt to save civilization we have decided to blow up the moon. Our plan is to divide the moon into many small chunks, and then, working inward from the surface, accelerate each chunk away from the center of the moon at escape velocity.

   **(a)** Use an integral to determine the energy necessary to blow up the moon in this way. (This quantity is called the *gravitational self-binding energy* of the moon.) Assume that the moon is a perfect sphere with radius 2 million meters and constant density 4000 kilograms per cubic meter. (Under this assumption, the energy necessary to accelerate a chunk of moon with mass $m$ to escape velocity, starting $\rho$ meters from the center of the moon, is approximately $\rho^2 m \times 10^{-6}$ Joules.) We only get one shot at this, and we've made a lot of simplifying assumptions, so round your answer **up** to **one** significant figure.

   **(b)** The world's combined nuclear arsenal is estimated as equivalent to about 5000 megatons of TNT. Is this enough to get the job done? (One ton of TNT is about $4 \times 10^9$ Joules.)

   **(c)** The sun outputs energy at a rate of approximately $4 \times 10^{26}$ Watts. Assume that we can focus all this energy on the moon. (This is called a *Nicoll-Dyson laser*.) How much time do we need? Answer in human-readable units.