Math 2163

Jeff Mermin's section, Test 2, October 25

On the essay questions (# 3–9) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

Do not evaluate any integrals on this test. If you would take an integral, instead simplify the integrand and the limits of integration (if any), and leave the integral as your final answer.

1. (**30 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   In the statements below, a, b, c, d, h, k, and L are numbers, $f = f(x, y)$ is a smooth function defined on some subset of $\mathbb{R}^2$ which includes (a, b) (but may or may not include (c, d)), $F = F(x, y, z)$ is a smooth function defined on $\mathbf{R}^3$, x, y, z, and t are variables, $\mathbf{r} = \mathbf{r}(t)$ and $\mathbf{s} = \mathbf{s}(t)$ are parametric curves in $\mathbf{R}^3$, and $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are vectors.

   (a) $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.

   (b) $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$.

   (c) If $\lim\limits_{(x,y)\to(c,d)} f(x, y) = L$, then $\lim\limits_{x\to c} f(x, d) = L$.

   (d) If (a, b) is a critical point of f, $f_{xx}(a, b) = 1$, and $f_{yy}(a, b) = -1$, then (a, b) is a saddle point of f.

   (e) If z is defined implicitly as a function of x and y by $F(x, y, z) = 0$, then $\dfrac{\partial z}{\partial x} = \dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$.

   (f) $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$.

   (g) $\dfrac{d}{dt}(\mathbf{r} \times \mathbf{s}) = \dfrac{d\mathbf{r}}{dt} \times \dfrac{d\mathbf{s}}{dt}$.

   (h) The tangent plane to the graph of the function $z = f(x, y)$ at the point $(a, b, f(a, b))$ is the graph of the linearization of f at (a, b).

   (i) If $\mathcal{L}$ is the linearization of f at (a, b) and $f_{xx} = f_{yy} = 0$, then $f(a + h, b + k) = \mathcal{L}(a + h, b + k)$.

   (j) $\mathbf{x} \bullet \mathbf{y} = \mathbf{y} \bullet \mathbf{x}$.

2. (**25 points**) Match the functions to the contour maps. No justification is necessary on this problem, but incorrect answers with explanation may earn partial credit.

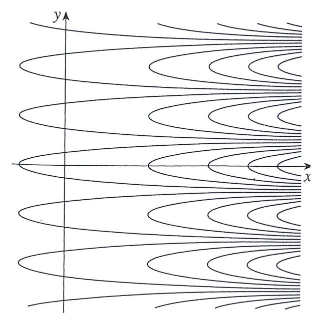
   (a) $z = \sin x - \sin y$

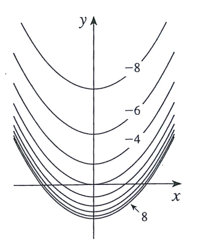
   (b) $z = e^x \cos y$

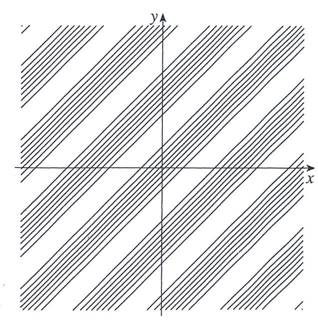
   (c) $z = (x^2 - y) + e^{x^2 - y} - 3$

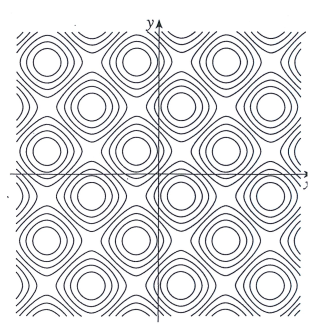
   (d) $z = \dfrac{x - y}{1 + x^2 + y^2}$

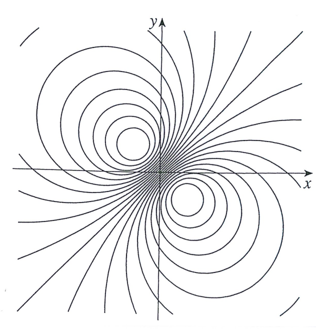
   (e) $z = \sin(y - x)$











3. (**30 points**) Let $f(x, y) = y^x$. (To avoid concerns about continuity, assume $x > 0$ and $y > 0$.) Find all four second partial derivatives of f.

4. (**15 points**) Let $f(x, y, z) = x^2 - 2y^2 + z^2$. If $\mathbf{v} = (2, 3, 6)$, find the unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$, and the directional derivative $D_{\mathbf{u}}(f)$.

5. (**15 points**) Let S be the surface defined by the equation $x^2 - 2y^2 + z^2 = 2$. Verify that $P = (3, 4, 5)$ is on S, and then find an equation for the tangent plane to S at P.

6. (**20 points**) Let $f(x, y) = \sin x \sin y$. You may assume that

$$f_x = \cos x \sin y \qquad f_y = \sin x \cos y$$

$$f_{xx} = -\sin x \sin y \qquad f_{xy} = \cos x \cos y \qquad f_{yy} = -\sin x \sin y$$

Decide whether the points below are critical points of f. Then, if they are, classify them as local maxima, local minima, or saddle points.

(a) $P = (0, 0)$

(b) $Q = (0, \frac{\pi}{2})$

(c) $R = (\frac{\pi}{2}, 0)$

(d) $S = (\frac{\pi}{2}, \frac{\pi}{2})$

7. (**15 points**) Prove that $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ does not exist, or provide strong evidence that it does.

8. (**Extra credit: 10 points**)

   Appian (*The Punic Wars*, chapter 1, paragraph 1) tells the following story of Dido's arrival in northern Africa, at the site of what would later become Carthage, with her band of Phoenician exiles (translation by Horace White, 1899):

   > Being repelled by the inhabitants, they asked for as much land for a dwelling place as they could encompass with an ox-hide. The Africans laughed at this frivolity of the Phoenicians and were ashamed to deny so small a request. Besides, they could not imagine how a town could be built in so narrow a space, and wishing to unravel the mystery they agreed to give it, and confirmed the promise by an oath.

   What should Dido do?

9. (**Extra credit: 10 points**)

   Describe how you would use techniques from Chapter 14 to solve the problem below. (If you would use algebra to solve equations, state what the equations would be and explain what you would do with the solutions, but don't actually solve.)

   > Find the maximum value of $f(x, y) = x^2y + x + y$ on the region D defined by the inequality $4x^2 + 5xy + 7y^2 \leq 24$. (You may assume D is closed and bounded.)