Math 2163

Section 2, Final exam, December 11

On the essay questions (# 2–13) write legibly in complete sentences, in such a way that I can easily tell what you are doing and why.

1. (**40 points**)Indicate whether the following statements are true or false. ("True" means "Always true", "false" means "sometimes false".) No justification is necessary on this problem. **Write the entire word "True" or "False"**. Illegible or abbreviated answers will receive no credit.

   a, b, c, d, d' and L are numbers. $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^3$. R is a region in $\mathbb{R}^2$ or $\mathbb{R}^3$. F, f and g are smooth functions on their domains, which include R. $C : \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a parametric curve. (u, v) is an alternative coordinate system on $\mathbb{R}^2$.

   (a) $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$.

   (b) If $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$,, then $\lim\limits_{(x,y)\to(a,b)} (\cos f(x, y)) = \cos L$.

   (c) If $R = \{(x, y, z) : x^2 + y^2 + z^2 \leq 9\}$, then $\iiint\limits_R dV = 36\pi$.

   (d) If R_1 and R_2 are disjoint regions in $\mathbb{R}^3$, and R is their union, then $\iiint\limits_R f(x, y, z)dV = \iiint\limits_{R_1} f(x, y, z)dV + \iiint\limits_{R_2} f(x, y, z)dV$.

   (e) $\iint\limits_R f(x, y)g(x, y)dA = \left(\iint\limits_R f(x, y)dA\right)\left(\iint\limits_R g(x, y)dA\right)$.

   (f) If R is the sphere of radius one about the origin, then $\iiint\limits_R f dV = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=1} f\rho^2 \sin\phi \, d\rho d\phi d\theta$.

   (g) There are eight possible orders of integration for a triple iterated integral.

   (h) The vector $\langle dx, dy, dz\rangle$ is normal to the graph of $z = f(x, y)$.

(i) The distance between the planes $G : ax + by + cz = d$ and $G' : ax + by + cz = d'$ is $|d - d'|$.

(j) $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 1.$

(k) $\frac{d\mathbf{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle.$

(l) If (a, b) is a critical point of f, $f_{xx}(a, b) = 1$, and $f_{yy}(a, b) = -1$, then (a, b) is a saddle point of f.

(m) The volume of a right circular cylinder of radius 1 and height 2 is $\int_{z=0}^{z=2} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} dr d\theta dz.$

(n) Every trace of a hyperboloid is a hyperbola.

(o) $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$.

(p) If f has two local maxima, then it must have a local minimum.

(q) $\frac{\partial}{\partial x}(f + g) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}.$

(r) $\iint\limits_R f\, dxdy = \iint\limits_R f\, dr d\theta.$

(s) $\nabla F(a, b, c)$ is normal to the surface $F(x, y, z) = 0$ at the point (a, b, c).

(t) If $\lim\limits_{x\to 0} f(x, b) = \lim\limits_{y\to 0} f(a, y) = L$, then $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L.$

2. (**10 points**) Let $\mathbf{a} = \langle 6, 10, -6 \rangle$ and $\mathbf{b} = \langle 1, -2, 2 \rangle$. Compute $\mathbf{a} \bullet \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$.

3. (**10 points**) Find an equation of the plane through the points $(4, -2, 1)$, $(4, -5, -3)$, and $(2, -4, -5)$.

4. (**10 points**) Determine whether or not the planes $F : 5x - 2y + 2z = -3$ and $G : 5x - 2y + 2z = -5$ intersect. If they do, find an equation for the line of intersection. If they do not, find the distance between F and G.

5. (**10 points**) Find the directional derivative $D_{\mathbf{u}}(f)$ where $f(x, y) = \ln(2x^2 - y)$ and $\mathbf{u} = \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$.

6. (**10 points**) Find the equation for the tangent line to the surface $C : \mathbf{r}(t) = \left\langle \ln t, t^2, e^t \right\rangle$ at the point $(0, 1, e)$.

7. (**10 points**) Consider the function $f(x, y) = x^5 - 2y^3 + y - 5xy - 11$. You do not have to compute the derivatives

$$f_x = 5x^4 - 5y, \qquad f_y = 1 - 6y^2 - 5x$$
$$f_{xx} = 20x^3, \qquad f_{xy} = -5, \qquad f_{yy} = -12y$$

Determine whether the points below are critical points of f. If they are, classify them as local maxima, local minima, or saddle points.

(a) $P = (0, 0)$.

(b) $Q = (-1, 1)$.

8. (**10 points**) Compute $\displaystyle\int_{x=0}^{x=1} \int_{y=1}^{y=2} e^y \, dydx$.

9. (**10 points**) Write $\displaystyle\iint\limits_R 12e^{x^2+y^2}\,dA$ as an iterated integral, where R is the region $R = \{x^2 + y^2 \leq 1,\ y \geq 0\}$. **Do not evaluate the integral.**

10. (**10 points**) Express $\displaystyle\int_C \langle x^2 + y^2, x^2 - y^2\rangle \bullet d\mathbf{r}$ in a form that a Calculus II student would understand, where C is the semicircle $x = \sqrt{1-y^2}$ pointing from $(0,-1)$ to $(0,1)$. (Calculus II students understand both numbers and simple definite integrals in one variable, so it is sufficient but **not necessary to evaluate the integral**).

11. (**10 points**) Compute $\int_C \mathbf{F} \bullet d\mathbf{r}$, where C is the line from $(0,0)$ to $(2,1)$ followed by the line from $(2,1)$ to $(3,3)$, and $\mathbf{F} = \langle e^y, xe^y \rangle$.

12. (**10 points**) Let $\mathbf{F} = \langle 2x+y, x+2y \rangle$. Is $\mathbf{F}$ conservative? If it is, find a function $f(x,y)$ such that $\mathbf{F} = \nabla f$.

13. (**Extra credit: 10 points**) Find the maximum and minimum values of the function $f(x, y) = (x - 1)^2 + y^2$ on the circle $x^2 + y^2 \leq 4$.

14. (**Extra credit: 10 points**) In the space remaining on this page, write down a sequence which goes to infinity as fast as possible. ($\{a_n\}$ goes to infinity faster than $\{b_n\}$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$.) You may express the sequences in any way you'd like, including using words or recursion rather than explicit formulas (and in fact you may need to use some English to define any nonstandard notation.) However, it should be possible (at least in theory) for me to determine the exact value of every term in your sequence without any reference to the real world.