A HOMOLOGICAL CHARACTERIZATION FOR FREENESS OF MULTI-ARRANGEMENTS

Abstract. We introduce a co-chain complex associated to a multi-arrangement and prove that its cohomologies determine freeness of the associated module of multi-derivations. The co-chain complex is constructed from modules introduced by Brandt and Terao to study k-formality. As a consequence, we prove that if $(\mathcal{A},\mathbf{m})$ is a free multi-arrangement then $\mathcal{A}$ is k-formal for all $k \geq 2$. We use this homological method to study freeness of multi-arrangements in moduli and certain classes of free arrangements whose restrictions are not free.

Contents

1. Introduction

A central hyperplane arrangement, which we will denote by $\mathcal{A}$, is a union of hyperplanes passing through the origin in a vector space $V \cong \mathbb{K}^\ell$, where $\mathbb{K}$ is a field. Write S for the symmetric algebra of V^*, which is isomorphic to a polynomial ring in ℓ variables. Then $\mathcal{A}$ is the union of the zero-locus of linear forms α_H, one for each hyperplane H in $\mathcal{A}$. The module of logarithmic $\mathcal{A}$-derivations, denoted $D(\mathcal{A})$, consists of derivations $\theta \in \mathrm{Der}_{\mathbb{K}}(S)$ satisfying $\theta(\alpha_H) \in \alpha_H S$ for every $H \in \mathcal{A}$. Study of this module was initiated by Saito [24]; it is of particular interest to know when $D(\mathcal{A})$ is a free S-module. In this case $\mathcal{A}$ is called a free arrangement.

Let $\mathbf{m} : \mathcal{A} \to \mathbb{Z}_{>0}$ be a function, called a multiplicity, associating to each hyperplane H a positive integer $\mathbf{m}(H)$; the pair $(\mathcal{A},\mathbf{m})$ is called a multi-arrangement. The module of derivations of $(\mathcal{A},\mathbf{m})$, denoted $D(\mathcal{A},\mathbf{m})$, consists of those derivations $\theta \in \mathrm{Der}_{\mathbb{K}}(S)$ satisfying $\theta(\alpha_H) \in \alpha_H^{\mathbf{m}(H)} S$ for every $H \in \mathcal{A}$. If $D(\mathcal{A},\mathbf{m})$ is a free S-module we say $(\mathcal{A},\mathbf{m})$ free and $\mathbf{m}$ is a free multiplicity of $\mathcal{A}$. Due to a criterion stated by Ziegler [40] and later improved by Yoshinaga [35], freeness of multi-arrangements is closely linked to freeness of arrangements.

There have been major advances in the understanding of multi-arrangements during the last decade. In particular, the characteristic polynomial has been defined for multi-arrangements by Abe, Terao, and Wakefield [6] and they show that Terao's factorization theorem holds for this characteristic polynomial. Moreover, the addition-deletion theorem has also been extended by Abe, Terao, and Wakefield to multi-arrangements [7]. This improved theory of multi-arrangements has recently led to remarkable progress in understanding freeness of arrangements [4, 1].

In this paper we add to the list of available tools for studying multi-arrangements by introducing a homological characterization for freeness. The characterization involves building a co-chain complex which we denote $\mathcal{D}^\bullet(\mathcal{A},\mathbf{m})$ from modules constructed by Brandt and Terao [12] to study k-formality (see Definition 3.5 for details). Chain complexes having very similar properties to $\mathcal{D}^\bullet(\mathcal{A},\mathbf{m})$ appear in the theory of algebraic splines [10, 27]; applying techniques of Schenck and Stiller [25, 28] yields our main result, stated below.

Theorem 1.1 (Homological characterization of freeness)**.** *The multi-arrangement $(\mathcal{A},\mathbf{m})$ is free if and only if $H^k(\mathcal{D}^\bullet(\mathcal{A},\mathbf{m}))=0$ for $k>0$. Moreover, $D(\mathcal{A},\mathbf{m})$ is locally free if and only if $H^k(\mathcal{D}^\bullet(\mathcal{A},\mathbf{m}))$ has finite length for all $k>0$.*

Weaker versions of this statement have been proved recently and used to classify free multiplicities on several rank three arrangements [15, 13, 14]. For simple arrangements, the forward direction of the first statement in Theorem 1.1 follows from work of Brandt and Terao [12]. Homological methods are not new in the study of freeness of arrangements; besides the aforementioned work of Brandt and Terao, Yuzvinsky developed and studied the theory of cohomology of sheaves of differentials on arrangement lattices to great effect in [37, 38, 39]. While we will not attempt to generalize this framework to multi-arrangements, Yuzvinsky's work, along with Brandt and Terao's, is an important motivation for this paper.

The remainder of the paper is devoted to applications of this homological criterion. In § 3 we extend a combinatorial bound on projective dimension of $D(\mathcal{A},\mathbf{m})$ due to Kung and Schenck in the case of simple arrangements. In § 4 we elucidate the connection to k-formality and use the homological characterization of Theorem 1.1 to extend a result of Brandt and Terao [12] to multi-arrangements in Corollary 4.10.

Following the initial applications of this homological characterization of freeness, we describe in § 5 how the chain complex $\mathcal{D}^\bullet(\mathcal{A},\mathbf{m})$ can be concretely computed. We have implemented this construction in the computer algebra system Macaulay2 [19]. The code for constructing the chain complex, as well as a file working through many of the examples in this paper, may be found on the author's website: math.okstate.edu/∼mdipasq. In § 5 we also explicitly work out the structure of $\mathcal{D}^\bullet(\mathcal{A},\mathbf{m})$ for graphic arrangements and show that Theorem 1.1 recovers the main result of [15].

In § 6, we study a class of arrangements which we call TF_2 arrangements; these are formal arrangements whose relations of length three are linearly independent. We believe this study is well-motivated by the interesting behavior of multi-TF_2 arrangements in moduli as well as additional counter-examples to Orlik's conjecture which arise in the process. We illustrate this in § 1.1 before proceeding to the body of the paper. If $\mathcal{A}$ is a TF_2 arrangement, freeness of $(\mathcal{A},\mathbf{m})$ is determined by the vanishing of the single cohomology module $H^1(\mathcal{D}^\bullet(\mathcal{A},\mathbf{m}))$, making these arrangements well-suited to the homological methods afforded by Theorem 1.1. We show that a TF_2 arrangement is free if and only if it is supersolvable. We completely classify free multiplicities on non-free TF_2 arrangements in Proposition 6.2 and Theorem 6.10. Moreover, we show that free multiplicities of free TF_2 arrangements can be determined in a combinatorial fashion from the exponents of its rank two sub-arrangements in Theorem 6.6.

We also give in § 7 a syzygetic criterion for freeness of a multi-arrangement of lines, generalizing a criterion for freeness of A_3 multi-arrangements from [13]. Specializing to simple line arrangements gives an equivalent formulation of Terao's question for line arrangements, phrased in terms of syzygies of a certain module presented by a matrix of linear forms (Question 7.4).

Acknowledgements: I am indebted to Stefan Tohaneanu for pointing out his paper [32], which provided the inspiration to generalize the homological arguments in [15]. The current work would not be possible without the collaboration of Chris Francisco, Jeff Mermin, Jay Schweig, and Max Wakefield on previous papers [13, 14]. Takuro Abe has been a consistent source of inspiring discussions and many patient explanations via e-mail. Computations in the computer algebra system Macaulay2 [19] were very useful at all stages of research.

1.1. **Examples.** In this section we illustrate results which can be obtained by applying the homological criterion for freeness (Theorem 1.1). The three examples in this section are TF_2 arrangements, the definition and analysis of which appears in § 6.

Example 1.2. Consider the line arrangement $\mathcal{A}(\alpha,\beta)$ defined by $xyz(x-\alpha z)(x-\beta z)(y-z)$ where $\alpha,\beta\in\mathbb{K}$. See Figure 1 for a projective picture of this arrangement over $\mathbb{R}$. Clearly if $\alpha\neq\beta,\alpha\neq 0$, and $\beta\neq 0$, then the intersection lattice $L(\mathcal{A}(\alpha,\beta))$ does not change. In fact, the arrangements $\mathcal{A}(\alpha,\beta)$ with $\alpha\neq\beta,\alpha\neq 0$, and $\beta\neq 0$ comprise the moduli space of this lattice (see Appendix A for a brief summary of the moduli space of a lattice). It is easily checked that $\mathcal{A}(\alpha,\beta)$ is supersolvable.

We will see in Theorem 6.6 that the freeness of the multi-arrangement $(\mathcal{A}(\alpha,\beta),\mathbf{m})$ can be determined if the exponents of the rank two sub multi-arrangements are known. Write $\mathbf{m}(x),\mathbf{m}(y),\ldots$ for the multiplicity assigned to, respectively, $x=0,y=0,\ldots$. There are two rank-two sub multi-arrangements of $(\mathcal{A}(\alpha,\beta),\mathbf{m})$ defined by

$$\begin{aligned}\tilde{X}_1 =&\quad y^{\mathbf{m}(y)}z^{\mathbf{m}(z)}(y-z)^{\mathbf{m}(y-z)} \text{ and}\\ \tilde{X}_2 =&\quad x^{\mathbf{m}(x)}z^{\mathbf{m}(z)}(x-\alpha z)^{\mathbf{m}(x-\alpha z)}(x-\beta z)^{\mathbf{m}(x-\beta z)}.\end{aligned}$$

In Example 6.8, we deduce from Theorem 6.6 that $(\mathcal{A}(\alpha,\beta),\mathbf{m})$ is free if and only if either $\tilde{X}_1$ or $\tilde{X}_2$ has $\mathbf{m}(z)$ as an exponent. This property is sensitive to the characteristic of $\mathbb{K}$; we will assume in the remainder of this example that $\mathbb{K}$ has characteristic zero.

Write $M_1=\mathbf{m}(y)+\mathbf{m}(z)+\mathbf{m}(y-z)$ and $M_2=\mathbf{m}(x)+\mathbf{m}(z)+\mathbf{m}(x-\alpha z)+\mathbf{m}(x-\beta z)$. If $\mathbb{K}$ has characteristic zero, the exponents of the multi-arrangement $\tilde{X}_1$ are known [33]; $\mathbf{m}(z)$ is an exponent if and only if $M_1\le 2\mathbf{m}(z)+1$. So we assume $M_1>2\mathbf{m}(z)+1$ and determine when $\tilde{X}_2$ has an exponent of $\mathbf{m}(z)$.

It is not difficult to show that if $\mathbf{m}(z)$ is an exponent of $\tilde{X}_2$, then $\mathbf{m}(z)=\max\{\mathbf{m}(x),\mathbf{m}(z),\mathbf{m}(x-\alpha z),\mathbf{m}(x-\beta z)\}$ (see Lemma B.2). From [34] it is known that $\mathbf{m}(z)$ is an exponent of $\tilde{X}_2$ if $M_2\le 2\mathbf{m}(z)+1$. Moreover it follows from [3, Theorem 1.6] that $\mathbf{m}(z)$ is not an exponent of $\tilde{X}_2$ if $M_2>2+2\mathbf{m}(z)$ (this also requires that $\mathbb{K}$ has characteristic zero). However if $M_2=2+2\mathbf{m}(z)$ then it is only

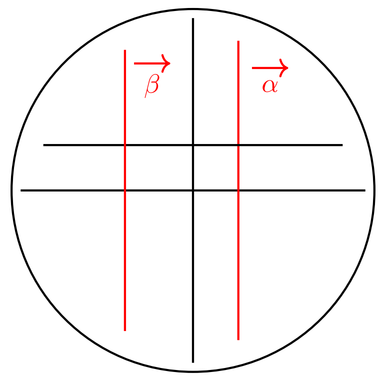

FIGURE 1. A projective picture emphasizing the moduli in Example 1.2

known that $\mathbf{m}(z)$ is not an exponent of $\tilde{X}_2$ for *generic* choices of α and β (at least if $\mathbb{K} = \mathbb{C}$ [34]).

To see what can happen if $M_2 = 2 + 2\mathbf{m}(z)$, consider the multi-arrangement $(\mathcal{A}(\alpha, \beta), \mathbf{m})$ defined by

$$x^3y^3z^3(x - \alpha z)(x - \beta z)(y - z)^3.$$

Then $\tilde{X}_1 = x^3y^3(y - z)^3$ and $\tilde{X}_2 = x^3z^3(x - \alpha z)(x - \beta z)$. The exponents of $\tilde{X}_1$ are $(4, 5)$, while the exponents of $\tilde{X}_2$ are $(4, 4)$ if $\alpha \neq -\beta$ and $(3, 5)$ if $\alpha = -\beta$ (see [40] or Lemma B.1). By Theorem 6.6, $(\mathcal{A}(\alpha, \beta), \mathbf{m})$ is free if and only if $\alpha = -\beta$.

As a consequence, we see that for a fixed multiplicity $\mathbf{m}$ the free multi-arrangements $(\mathcal{A}, \mathbf{m})$ in the moduli space of $L(\mathcal{A})$ can form a non-empty proper Zariski closed subset, even when $\mathcal{A}$ is supersolvable over a field of characteristic zero. In contrast, Yuzvinsky has shown that free arrangements form a Zariski open subset of the moduli space of $L(\mathcal{A})$ [39].

Example 1.3. Let $\mathcal{A}(\alpha, \beta)$ be the arrangement with defining polynomial $\mathcal{Q}(\mathcal{A}(\alpha, \beta)) = xyz(x-\alpha y)(x-\beta y)(y-z)(x-z)$, where $\alpha, \beta \in \mathbb{K}$. See Figure 2 for a projective drawing of this arrangement over $\mathbb{R}$. It is straightforward to show that if $\alpha \neq 1, \beta \neq 1$, and $\alpha \neq \beta$, then the lattice $L(\mathcal{A}(\alpha, \beta))$ does not change. Just as in Example 1.2, these arrangements comprise the moduli space of this lattice. It is easily checked that $\mathcal{A}(\alpha, \beta)$ is not free for any choice of α, β since its characteristic polynomial does not factor.

We will see in Theorem 6.10 that if $\mathbb{K}$ has characteristic 0, the multi-arrangement $(\mathcal{A}(\alpha, \beta), \mathbf{m})$ is free if and only if its defining equation has the form

$$\mathcal{Q}(\mathcal{A}, \mathbf{m}) = x^ny^nz^n(x - \alpha y)(x - \beta y)(y - z)(x - z),$$

where $n > 1$ is an integer and $\alpha^{n-1} = \beta^{n-1} \neq 1$. In particular, if α/β is not a root of unity in $\mathbb{K}$, then $\mathcal{A}$ is totally non-free, meaning it does not admit any free multiplicities. For instance, if $\mathbb{K} = \mathbb{R}$, then $\mathcal{A}$ admits a free multiplicity if and only if $\alpha = -\beta$ (precisely when $n > 1$ is odd). Since the arrangements $\mathcal{A}(\alpha, \beta)$ with $\alpha \neq 1, \beta \neq 1$, and $\alpha \neq \beta$ all have the same intersection lattice, this shows that the property of being totally non-free is not combinatorial. In contrast, Abe, Terao, and Yoshinaga have shown that the property of being totally free is combinatorial [8].

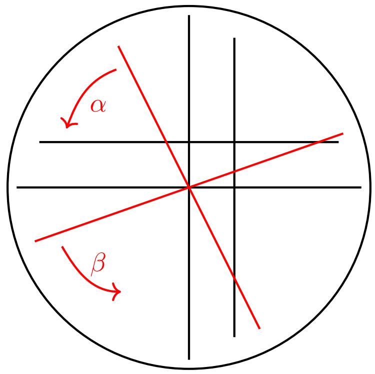

FIGURE 2. A projective picture emphasizing the moduli in Example 1.3

Example 1.4. Let $S = \mathbb{K}[x_0, \dots, x_r]$ and let $\mathcal{A} \subset \mathbb{K}^{r+1}$ be the arrangement defined by

$$\mathcal{Q}(\mathcal{A}) = x_0 \left(\prod_{i=1}^{r} (x_i^2 - x_0^2) \right) (x_1 - x_2) \cdots (x_{r-1} - x_r)(x_r + x_1)$$

Let H be the hyperplane defined by x_0. In Proposition 6.12, we will show that $\mathcal{A}$ is free using Yoshinaga's theorem [35] and Theorem 6.10. Moreover, we will prove that $\operatorname{pdim}(D(\mathcal{A}^H)) = r - 3$, the largest possible. In fact, we will show more: the minimal free resolution of $D(\mathcal{A}^H)$ is a truncated and shifted Koszul complex, so it is linear. As with the previous two examples, the key to our analysis is that the restriction $\mathcal{A}^H$ is a TF_2 arrangement, which is particularly well suited to the homological methods we introduce in this paper.

This family of examples is interesting because it adds to a short list of arrangements known to fail Orlik's conjecture. This conjecture states that $\mathcal{A}^H$ is free whenever $\mathcal{A}$ is free [22]. The only counterexamples to this conjecture of which we are aware appear in work of Edelman and Reiner [16, 17]. For the small ranks that we have been able to compute, our examples differ from theirs in that $D(\mathcal{A}^H)$ for the examples of Edelman and Reiner seems to be always 'almost free' - that is $D(\mathcal{A}^H)$ has only one more generator than the rank of $\mathcal{A}^H$ and there is only a single relation among these generators. This latter behavior has been studied in a recent article of Abe [1].

2. PRELIMINARIES

Fix a field $\mathbb{K}$, let V be a $\mathbb{K}$-vector space of dimension ℓ, and V^* the dual vector space. Set $S = \operatorname{Sym}(V^*)$, the symmetric algebra on V^*. A hyperplane arrangement $\mathcal{A} \subset V$ is a union of hyperplanes H defined by the vanishing of the affine linear form $\alpha_H \in V^*$; the *defining polynomial* of $\mathcal{A}$ is $\mathcal{Q}(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$. We will consistently abuse notation and write $H \in \mathcal{A}$ if H is one of the hyperplanes whose union forms $\mathcal{A}$. Moreover, we will write $|\mathcal{A}|$ for the number of hyperplanes in $\mathcal{A}$.

The *rank* of a hyperplane arrangement $\mathcal{A} \subset V$ is $r = r(\mathcal{A}) := \dim V - \dim(\cap_{H \in \mathcal{A}} H)$. The arrangement $\mathcal{A} \subset V$ is called *essential* if $r(\mathcal{A}) = \dim V$ and *central* if $\cap_i H_i \neq \emptyset$. We will always assume $\mathcal{A}$ is a central hyperplane arrangement. We refer the reader to the landmark book of Orlik and Terao [23] for further details on arrangements.

The intersection lattice $L = L(\mathcal{A})$ of $\mathcal{A}$ is the lattice whose elements (flats) are all possible intersections of the hyperplanes of $\mathcal{A}$, ordered with respect to reverse inclusion. We will use $<$ to denote the ordering on the lattice, so if $X, Y \in L(\mathcal{A})$ and $X \subseteq Y$ as intersections, then $Y \leq X$ in $L(\mathcal{A})$. This is a ranked lattice with rank function the codimension of the flat; we denote by $L_i = L_i(\mathcal{A})$ the flats $X \in L(\mathcal{A})$ with rank i. Given a flat $X \in L(\mathcal{A})$, the (closed) subarrangement $\mathcal{A}_X$ is the hyperplane arrangement of those hyperplanes of $\mathcal{A}$ which contain X, and the *restriction* of $\mathcal{A}$ to X, denoted $\mathcal{A}^X$, is the hyperplane arrangement (in linear space corresponding to X) with hyperplanes $\{H \cap X : H \not< X \text{ in } L(\mathcal{A})\}$. If $X < Y$, the interval $[X, Y] \subset L(\mathcal{A})$ is the sub-lattice of all flats $Z \in L$ so that $X \leq Z \leq Y$. This is the intersection lattice of the arrangement $\mathcal{A}_X^Y$.

If $\mathcal{A} \subset V_1$ and $\mathcal{B} \subset V_2$ are two arrangements, then the product of $\mathcal{A}$ and $\mathcal{B}$ is the arrangement

$$\mathcal{A} \times \mathcal{B} = \{H \oplus V_2 : H \in \mathcal{A}\} \cup \{V_1 \oplus H' : H' \in \mathcal{B}\},$$

and the arrangements $\mathcal{A}, \mathcal{B}$ are *factors* of $\mathcal{A} \times \mathcal{B}$. If an arrangement can be written as a product of two arrangements we say it is *reducible*, otherwise we call it *irreducible*. (Notice that an arrangement is not essential if and only if it has the empty arrangement as a factor).

If $\mathcal{A} \subset V$ is an arrangement the module of derivations of $\mathcal{A}$, denoted $D(\mathcal{A})$, is defined by

$$D(\mathcal{A}) = \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) | \theta(\alpha_H) \in \langle \alpha_H \rangle \text{ for all } H \in \mathcal{A}\}.$$

If $D(\mathcal{A})$ is free as an S-module, we say $\mathcal{A}$ is free.

Definition 2.1. A multi-arrangement $(\mathcal{A}, \mathbf{m})$ is an arrangement $\mathcal{A} \subset V$, along with a function $\mathbf{m} : \mathcal{A} \to \mathbb{Z}_{>0}$ assigning a positive integer to every hyperplane. The *defining polynomial* of a multi-arrangement $(\mathcal{A}, \mathbf{m})$ is $\mathcal{Q}(\mathcal{A}, \mathbf{m}) := \prod_{H \in \mathcal{A}} \alpha_H^{\mathbf{m}(H)}$. The module of multi-derivations $D(\mathcal{A}, \mathbf{m})$ is

$$D(\mathcal{A}, \mathbf{m}) = \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) | \theta(\alpha_H) \in \langle \alpha_H^{\mathbf{m}(H)} \rangle \text{ for all } H \in \mathcal{A}\}$$

Lemma 2.2. *Let $(\mathcal{A}, \mathbf{m})$ be a multi-arrangement in $V \cong \mathbb{K}^\ell$. Let α_i be the form defining the hyperplane H_i, and set $m_i = \mathbf{m}(H_i)$. The module $D(\mathcal{A}, \mathbf{m})$ of multi-derivations on $\mathcal{A}$ is isomorphic to the kernel of the map*

$$\psi : S^{\ell+d} \to S^d,$$

where ψ is the matrix

$$\left(\begin{array}{c|ccc} & \alpha_1^{m_1} & & \\ B & & \ddots & \\ & & & \alpha_k^{m_k} \end{array}\right)$$

and B is the matrix with entry $B_{ij} = a_{ij}$, where $\alpha_j = \sum_{i,j} a_{ij} x_i$.

Proof. See the comments preceding [11, Theorem 4.6]. □

If $D(\mathcal{A}, \mathbf{m})$ is free as an S-module then we say that the multi-arrangement $(\mathcal{A}, \mathbf{m})$ is free and $\mathbf{m}$ is a *free multiplicity* of $\mathcal{A}$. If $D(\mathcal{A}, \mathbf{m})$ is free there is (by definition) a *basis* of derivations $\theta_1, \ldots, \theta_\ell \in D(\mathcal{A}, \mathbf{m})$ so that every other $\theta \in D(\mathcal{A}, \mathbf{m})$ can be written uniquely as a polynomial combination of $\theta_1, \ldots, \theta_\ell$. If $\mathcal{A}$ is central (which we will assume throughout), we may assume these derivations are homogeneous with degrees $d_i = \deg(\theta_i)$. The set $(d_1, \ldots, d_\ell)$ are called the *exponents* of $D(\mathcal{A}, \mathbf{m})$. We will always assume $d_1 \geq d_2 \geq \cdots \geq d_\ell$. Write $|\mathbf{m}|$ for $\sum_{H\in\mathcal{A}} \mathbf{m}(H)$. It follows from Saito's criterion (below) that if $D(\mathcal{A},\mathbf{m})$ is free with exponents $(d_1,\dots,d_\ell)$ then $\sum_{i=1}^{\ell} d_i = |\mathbf{m}|$.

Proposition 2.3 (Saito's criterion)**.** *Let $(\mathcal{A},\mathbf{m})$ be a central arrangement in a vector space V of dimension ℓ, and write $\mathbb{K}[x_1,\dots,x_\ell]$ for $Sym(V^*)$. Suppose $\theta_1,\dots,\theta_\ell$ are derivations with $\theta_i = \sum_{j=1}^{\ell} \theta_{ij}\frac{\partial}{\partial x_i}$. Write $M = M(\theta_1,\dots,\theta_\ell)$ for the $\ell\times\ell$ matrix of coefficients $M_{ij} = \theta_{ij}$. Then $D(\mathcal{A},\mathbf{m})$ is free with basis $\theta_1,\dots,\theta_\ell$ if and only if* $\det(M)$ *is a scalar multiple of the defining polynomial $\mathcal{Q}(\mathcal{A},\mathbf{m})$.*

If $X \in L(\mathcal{A})$, we write $(\mathcal{A}_X,\mathbf{m}_X)$ for the multi-arrangement $\mathcal{A}_X$ with multiplicity function $\mathbf{m}_X = \mathbf{m}|_{\mathcal{A}_X}$. If $(\mathcal{A}_X,\mathbf{m}_X)$ is free for every $X \neq \cap_{H\in\mathcal{A}} H \in L$, then we say $(\mathcal{A},\mathbf{m})$ is *locally free*; equivalently the associated sheaf $\widetilde{D(\mathcal{A},\mathbf{m})}$ is a vector bundle on $\mathbb{P}^{\ell-1}$.

Proposition 2.4. [5, Proposition 1.7] *Let $(\mathcal{A},\mathbf{m})$ be a multi-arrangement, $X \in L(\mathcal{A})$, and $(\mathcal{A}_X,\mathbf{m}_X)$ the corresponding closed subarrangement with restricted multiplicities. Then $pdim(D(\mathcal{A},\mathbf{m})) \geq pdim(D(\mathcal{A}_X,\mathbf{m}_X))$.*

Lemma 2.5 (Ziegler [40])**.** *For any arrangement $\mathcal{A} \subset V$, $pdim(D(\mathcal{A},\mathbf{m})) \leq r(\mathcal{A}) - 2$. In particular, if $r(\mathcal{A}) \leq 2$ then $(\mathcal{A},\mathbf{m})$ is free.*

If $\mathcal{A}$ is an arrangement and $H \in \mathcal{A}$, we denote by $(\mathcal{A}^H,\mathbf{m})$ the *Ziegler restriction* of $\mathcal{A}$ to $\mathcal{A}^H$; this is the arrangement $\mathcal{A}^H$ with the multiplicity function $\mathbf{m}^H$ defined by

$$\mathbf{m}^H(X) = \#\{H' \in \mathcal{A} : H' \cap H = X\}$$

for every $X \in \mathcal{A}^H$. We include the following criterion for freeness which is due to Yoshinaga [35]; the observation that we can restrict to codimension three was made in [9, Theorem 4.1].

Theorem 2.6. [35, Theorem 2.2] *An arrangement $\mathcal{A}$ over a field of characteristic zero is free if and only if, for some $H \in \mathcal{A}$:*

1. *$(\mathcal{A}^H,\mathbf{m}^H)$ is free and*
2. *$\mathcal{A}_X$ is free for every $X \neq 0 \in L_3(\mathcal{A})$ so that $H < X$.*

The second condition is sometimes stated as '$\mathcal{A}$ is locally free along H in codimension three.'

3. The homological criterion

Let $(\mathcal{A},\mathbf{m})$ be a multi-arrangement. In this section we prove Theorem 1.1; we describe the chain complex $\mathcal{D}^\bullet(\mathcal{A},\mathbf{m})$ and prove that $(\mathcal{A},\mathbf{m})$ is free if and only if $H^i(\mathcal{D}^\bullet(\mathcal{A},\mathbf{m})) = 0$ for all $i > 0$. The construction of the modules which comprise $\mathcal{D}^\bullet(\mathcal{A},\mathbf{m})$ is due to Brandt and Terao if $\mathbf{m} \equiv 1$ [31, 12]; we make the straightforward observation that the same definitions work also for multi-arrangements. We follow the presentation given in [12].

Definition 3.1. Set $D_0(\mathcal{A},\mathbf{m}) = D(\mathcal{A},\mathbf{m})$ and for $1 \leq k \leq r = r(\mathcal{A})$ inductively define $D_k(\mathcal{A},\mathbf{m})$ and $K_k(\mathcal{A},\mathbf{m})$ as the cokernel and kernel, respectively of the map

$$\tau_{k-1} = \tau_{k-1}(\mathcal{A}) : D_{k-1}(\mathcal{A},\mathbf{m}) \to \bigoplus_{X\in L_{k-1}} D_{k-1}(\mathcal{A}_X,\mathbf{m}_X),$$

where τ_k is a sum of maps $\phi_k(Y) : D_k(\mathcal{A}, \mathbf{m}) \to D_k(\mathcal{A}_Y, \mathbf{m}_Y)$. For $Y \in L$ with $r(Y) \geq k$, $\phi_k(Y)$ is defined inductively (the map for $k = 0$ is the usual inclusion of derivations) via the diagram in Figure 3: The center vertical map is projection, the

$$\begin{tikzcd}
D_{k-1}(\mathcal{A},\mathbf{m}) \arrow[r, "\tau_{k-1}(\mathcal{A})"] \arrow[d, "\phi_{k-1}(Y)"] & \bigoplus\limits_{X\in L_{k-1}} D_{k-1}(\mathcal{A}_X,\mathbf{m}_X) \arrow[r] \arrow[d, "p_{k-1}(Y)"] & D_k(\mathcal{A},\mathbf{m}) \arrow[r] \arrow[d, "\phi_k(Y)"] & 0 \\
D_{k-1}(\mathcal{A}_Y,\mathbf{m}_Y) \arrow[r, "\tau_{k-1}(\mathcal{A}_Y)"] & \bigoplus\limits_{\substack{X\leq Y\\ r(X)=k-1}} D_{k-1}((\mathcal{A}_Y)_X,(\mathbf{m}_Y)_X) \arrow[r] & D_k(\mathcal{A}_Y,\mathbf{m}_Y) \arrow[r] & 0
\end{tikzcd}$$

FIGURE 3. Diagram for Definition 3.1

left-hand square commutes, so $\phi_k(Y)$ may be defined so that the right-hand square commutes.

Remark 3.2. Given an arrangement $\mathcal{A}$, the only flat of L with rank 0 is V, the ambient space of $\mathcal{A}$. The module $D_1(\mathcal{A}, \mathbf{m})$ is the cokernel of the map

$$D_0(\mathcal{A},\mathbf{m}) \xrightarrow{\tau_0} \bigoplus_{X\in L_0} D_0(\mathcal{A}_X,\mathbf{m}),$$

in other words the cokernel of the inclusion

$$D(\mathcal{A},\mathbf{m}) \to D(V) = \mathrm{Der}_{\mathbb{K}}(S) \cong S^\ell,$$

where $\ell = \dim(V)$.

Remark 3.3. Fix a basis $x_1, \ldots, x_\ell$ for $S_1 = \mathrm{Sym}(V^*)_1$ and denote the corresponding basis of $\mathrm{Der}_{\mathbb{K}}(S)$ by $\partial_i = \partial/\partial x_i$. Number the hyperplanes of $\mathcal{A}$ by $H_1, \ldots, H_k$. Assume $H_j = V(\alpha_j)$, where $\alpha_j = \alpha_{H_j} = \sum_i a_{ij}x_i$. For some $H = H_j \in \mathcal{A}$ let $\partial_H = \sum_i a_{ij}\partial_i$.

For $H \in \mathcal{A}$, let $J(H) = \langle \alpha_H^{\mathbf{m}(H)} \rangle$, the ideal generated by $\alpha_H^{\mathbf{m}(H)}$ in S. Then $D(\mathcal{A}_H, \mathbf{m}_H) \subset \mathrm{Der}_{\mathbb{K}}(S)$ is isomorphic to $J(H)\partial_H \oplus S^{\ell-1}$, where $\ell = \dim(V)$ and $J(H)\partial_H$ denotes that $J(H)$ is living inside of the copy of S corresponding to the basis element ∂_H. So $D_1(\mathcal{A}_H, \mathbf{m}_H)$ is the cokernel of the inclusion $D(\mathcal{A}_H, \mathbf{m}_H) \to \mathrm{Der}_{\mathbb{K}}(S) \cong S^\ell$, which may be identified as $S\partial_H/J(H)$. There is then a natural map

$$\mathrm{Der}_{\mathbb{K}}(S) \cong S^\ell \xrightarrow{B} \bigoplus_{H\in\mathcal{A}} \frac{S}{J(H)} = \bigoplus_{X\in L_1} D_1(\mathcal{A}_X,\mathbf{m}_X),$$

where B is the matrix with entries $B_{ij} = a_{ij}$. The kernel of this map is $D(\mathcal{A}, \mathbf{m})$, its image is $D_1(\mathcal{A}, \mathbf{m})$, and its cokernel is $D_2(\mathcal{A}, \mathbf{m})$.

Remark 3.4. We will discuss computations of $D_k(\mathcal{A}, \mathbf{m})$ further in § 5.

Extending Remark 3.3, we assemble the modules $\bigoplus\limits_{X\in L_k} D_k(\mathcal{A}_X, \mathbf{m}_X)$ into a chain complex.

Definition 3.5. Set $\mathcal{D}^k = \bigoplus\limits_{X\in L_k} D_k(\mathcal{A}_X, \mathbf{m}_X)$. Define $\delta^k : \mathcal{D}^k \to \mathcal{D}^{k+1}$ by the composition $\mathcal{D}^k \to D_{k+1}(\mathcal{A}, \mathbf{m}) \xrightarrow{\tau_{k+1}} \mathcal{D}^{k+1}$, where the first map is the natural surjection from Definition 3.1. The derivation complex $\mathcal{D}^\bullet = \mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})$ is the chain

$$
\begin{tikzcd}
 & & 0 \arrow[dr] & & & \\
 & & & K_{i+1}(\mathcal{A},\mathbf{m}) \arrow[dr] & & 0 \\
 & & & & D_{i+1}(\mathcal{A},\mathbf{m}) \arrow[ur] \arrow[dr, "\tau_{i+1}(\mathcal{A})"] & \\
 & \mathcal{D}^{i-1} \arrow[rr, "\delta^{i-1}"] \arrow[dr] & & \mathcal{D}^{i} \arrow[ur] \arrow[rr, "\delta^{i}"] & & \mathcal{D}^{i+1} \\
 & & D_i(\mathcal{A},\mathbf{m}) \arrow[ur, "\tau_i(\mathcal{A})"'] \arrow[dr] & & & \\
 & K_i(\mathcal{A},\mathbf{m}) \arrow[ur] & & 0 & & \\
0 \arrow[ur] & & & & &
\end{tikzcd}
$$

FIGURE 4. Components of Definition 3.5

complex with modules $\mathcal{D}^k$ for $k = 0, \dots, r(\mathcal{A})$ and maps $\delta^k : \mathcal{D}^k \to \mathcal{D}^{k+1}$ for $k = 0, \dots, r(\mathcal{A}) - 1$.

Remark 3.6. The derivation complex $\mathcal{D}^\bullet$ is tautologically a complex from the definitions of $D_k(\mathcal{A}, \mathbf{m})$ and δ^k. The commutative diagram in Figure 4 shows how all the definitions so far fit together. Note that $K_i(\mathcal{A}, \mathbf{m})$ from Definition 3.1 may be identified with $H^i(\mathcal{D}^\bullet)$.

Remark 3.7. The chain complex $\mathcal{D}^\bullet$ in Definition 3.5 is essentially dual to a chain complex described in [32]; we will describe the precise connection in § 4.

Lemma 3.8. *For a multi-arrangement* $(\mathcal{A}, \mathbf{m})$*,* $H^0(\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})) \cong D(\mathcal{A}, \mathbf{m})$*.*

Proof. This is immediate from Remark 3.3. □

Now we proceed to the proof of Theorem 1.1. We use a few preliminary results.

Lemma 3.9. [12, Lemma 4.12] *For any* k*, the functors* $X \to D_k(\mathcal{A}_X, \mathbf{m}_X)$ *for* $X \in L$ *are local in the sense of* [29, Definition 6.4]*. Namely let* $P \in Spec(S)$*,* $X \in L$*, and set* $X(P) = \bigcap_{\substack{H \in \mathcal{A}_X \\ \alpha_H \in P}} H$*. Then*

- $D_k(\mathcal{A}_X, \mathbf{m}_X)_P = D_k(\mathcal{A}_{X(P)}, \mathbf{m}_{X(P)})_P$ *and*
- $\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})_P = \mathcal{D}^\bullet(\mathcal{A}_{X(P)}, \mathbf{m}_{X(P)})_P$*.*

Proof. For the first bullet, use the fact that $X \to D(\mathcal{A}_X, \mathbf{m})$ is local, the short exact sequences in Definition 3.1, and the fact that localization is an exact functor. The second bullet follows from the first. □

Proposition 3.10. *Let* $X \in L_k$ *and* $I(X) \subset S$ *denote the ideal generated by the linear forms* α_H *for all* $H \leq X$*. Then* $D_k(\mathcal{A}_X, \mathbf{m}_X)$ *is Cohen-Macaulay of codimension* k *and* $I(X)$ *is its only associated prime.*

Remark 3.11. Proposition 3.10 is implicit in the proof of [12, Proposition 4.13]; we provide a proof for completeness.

Proof. As usual, set $\ell = \dim(V)$. By changing coordinates, we may assume $X = V(x_1, \dots, x_k)$. The result is clear if $k = 0$ or $k = 1$, so we assume $k \geq 2$. Let $\pi_X : V \to X^{\perp} = W$ be the projection with center X and set $R = \mathrm{Sym}(W^*) \cong \mathbb{K}[x_{k+1}, \dots, x_\ell]$. Then we observe that

- $\mathcal{A}^\pi = \pi_X(\mathcal{A}_X)$ is an essential arrangement in W of rank $\ell - k = \dim W$,
- $D_k(\mathcal{A}^\pi, \mathbf{m}_X) \otimes_R S = D_k(\mathcal{A}_X, \mathbf{m}_X)$,
- $x_{k+1}, \dots, x_\ell$ is a regular sequence on $D_k(\mathcal{A}_X, \mathbf{m}_X)$,
- $D_k(\mathcal{A}_X, \mathbf{m}_X)/\langle x_{k+1}, \dots, x_\ell \rangle D_k(\mathcal{A}_X, \mathbf{m}_X) \cong D_k(\mathcal{A}^\pi, \mathbf{m}_X)$,
- and $\mathrm{Ass}(D_k(\mathcal{A}_X, \mathbf{m}_X)) = \{PS | P \in \mathrm{Ass}(D_k(\mathcal{A}^\pi, \mathbf{m}_X))\}$,

where the final bullet point follows from [21, Theorem 23.2], which describes behavior of associated primes under flat extensions. Hence it suffices to show that the only associated prime of $D_k(\mathcal{A}, \mathbf{m})$ when $k = r(\mathcal{A}) = \dim V$ is the maximal ideal of S. Consider the short exact sequence

$$0 \to D_{k-1}(\mathcal{A}, \mathbf{m}) \to \mathcal{D}^{k-1} = \bigoplus_{X \in L_{k-1}} D_{k-1}(\mathcal{A}_X, \mathbf{m}_X) \to D_k(\mathcal{A}, \mathbf{m}) \to 0$$

from Definition 3.1, and localize at a prime $P \in \mathrm{Spec}(S)$. If $\mathrm{codim}(P) \leq k-1$, then by induction either $\mathcal{D}^{k-1}_P$ vanishes (in which case $D_k(\mathcal{A}, \mathbf{m})_P = 0$) or $P = I(X)$ for some $X \in L$ of codimension $k-1$ and $\mathcal{D}^{k-1}_P = D_{k-1}(\mathcal{A}_X, \mathbf{m}_X)_P$. In the latter case, localizing the exact sequence above at $P = I(X)$ and using Lemma 3.9 yields the exact sequence

$$0 \to D_{k-1}(\mathcal{A}_X, \mathbf{m}_X)_{I(X)} \to D_{k-1}(\mathcal{A}_X, \mathbf{m}_X)_{I(X)} \to D_k(\mathcal{A}, \mathbf{m})_{I(X)} \to 0,$$

so clearly $D_k(\mathcal{A}, \mathbf{m})_{I(X)} = 0$. Hence the only prime in the support of $D_k(\mathcal{A}, \mathbf{m})$ is the homogeneous maximal ideal. □

Proof of Theorem 1.1. By Lemma 3.8, $D(\mathcal{A}, \mathbf{m}) \cong H^0(\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m}))$. Now we use the following result of Schenck and Stiller (see also [25]).

Theorem 3.12. [28, Theorem 3.4] *Suppose $C^\bullet = 0 \to C^0 \to C^1 \to C^2 \to \cdots \to C^t \to 0$ is a complex of $S = \mathbb{K}[x_1, \dots, x_\ell]$-modules so that, for $k = 0, \dots t$,*

- *C^k is Cohen-Macaulay of codimension k*
- *$H^k(\mathcal{C}^\bullet)$ is supported in codimension $\geq k+2$.*

Then $H^0(C^\bullet)$ is free if and only if $H^k(C^\bullet) = 0$ for $k > 0$ and locally free if and only if $H^k(C^\bullet)$ has finite length for $k > 0$.

By Proposition 3.10, $\mathcal{D}^k = \mathcal{D}^k(\mathcal{A}, \mathbf{m})$ is Cohen-Macaulay of codimension k. So we need to show that $H^k(\mathcal{D}^\bullet)$ is supported in codimension at least $k+2$. We use the fact that taking homology commutes with localization. So let P be a prime and consider the localized complex

$$\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})_P = \cdots \to \mathcal{D}^{k-1}_P \xrightarrow{\delta^{k-1}_P} \mathcal{D}^k_P \xrightarrow{\delta^k_P} \mathcal{D}^{k+1}_P \to \cdots$$

If $\mathrm{codim}(P) \leq k$, then we have seen in the proof of Proposition 3.10 that the localized map δ^{k-1}_P becomes an isomorphism, hence $H^k(\mathcal{D}^\bullet)_P = H^k(\mathcal{D}^\bullet_P) = 0$. Now suppose $\mathrm{codim}(P) = k+1$. If $P \neq I(X)$ for some $X \in L$ of codimension $k+1$, then let $X \in L_i$ ($i \leq k$) be the flat of maximal rank so that $I(X) \subset P$. If $r(X) \leq k-1$ then $H^k(\mathcal{D}^\bullet_P) = 0$ by Proposition 3.10. So suppose X has codimension

k. Then the localized map δ_P^{k-1} becomes an isomorphism again as in the proof of Proposition 3.10.

Finally suppose $P = I(X)$ for some $X \in L_{k+1}$. Localizing yields

$$\bigoplus_{\substack{Y \geq X \\ r(Y)=k-1}} D_{k-1}(\mathcal{A}_Y, \mathbf{m}_Y)_P \xrightarrow{\delta_P^{k-1}} \bigoplus_{\substack{Z \geq X \\ r(Z)=k}} D_k(\mathcal{A}_Z, \mathbf{m}_Z)_P \xrightarrow{\delta_P^k} D_{k+1}(\mathcal{A}_X, \mathbf{m}_X)_P.$$

By definition δ^{k-1} factors through $D_k(\mathcal{A}, \mathbf{m})$. Hence $H^k(\mathcal{D}^\bullet)_P$ is the middle homology of the three term complex

$$0 \to D_k(\mathcal{A}_X, \mathbf{m}_X)_P \xrightarrow{(\tau_k)_P} \bigoplus_{\substack{Z \geq X \\ r(Z)=k}} D_k(\mathcal{A}_Z, \mathbf{m}_Z)_P \xrightarrow{\delta_P^k} D_{k+1}(\mathcal{A}_X, \mathbf{m}_X)_P \to 0,$$

which is exact by Definition 3.1. It follows that $H^k(\mathcal{D}^\bullet)$ is supported in codimension $\geq k+2$. □

Remark 3.13. In the case of a simple arrangement, the forward implication of Theorem 1.1 follows from [12, Proposition 4.13].

Theorem 3.12 arises from a studying the hyperExt modules of $\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})$. Without the vanishing assumptions we may obtain the following.

Proposition 3.14. *Set $p_i = pdim(H^i(\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})))$ for $i > 0$. Then*

$$pdim(D(\mathcal{A}, \mathbf{m})) \leq \max_{i>0}\{p_i - i - 1\},$$

with equality if there is a single $i > 0$ for which $H^i(\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})) \neq 0$.

Proof. See [25, Lemma 4.11] or [15, § 3]. □

3.1. **A combinatorial bound on projective dimension.** We close this section by extending a combinatorial bound on projective dimension due to Kung and Schenck for simple arrangements [20, Corollary 2.3]. Recall that a generic arrangement of rank ℓ is one in which the intersection of every subset of $k \leq \ell$ hyperplanes has codimension k.

Corollary 3.15. *Let $(\mathcal{A}, \mathbf{m})$ be a multi-arrangement. If $\mathcal{A}_X$ is generic with $|\mathcal{A}_X| > r(X)$, then $pdim(D(\mathcal{A}, \mathbf{m})) \geq r(X) - 2$. In particular, if the matroid of $\mathcal{A}$ has a closed circuit of length m, then $pdim(D(\mathcal{A}, \mathbf{m})) \geq m - 3$.*

Proof. If $r(\mathcal{A}) = 2$ the statement is trivial so we will assume $r(\mathcal{A}) > 2$. Suppose $\mathcal{A}_X$ is generic with $|\mathcal{A}_X| > r(X)$. By Proposition 2.4, it suffices to show that $\mathrm{pdim}(D(\mathcal{A}_X, \mathbf{m}_X)) \geq r(X) - 2$. So we assume $\mathcal{A} = \mathcal{A}_X$ is essential and generic of rank r with $|\mathcal{A}| > r$ and prove $\mathrm{pdim}(D(\mathcal{A}, \mathbf{m})) = r - 2$.

In this case we claim the chain complex $\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})$ has the form $S^r \xrightarrow{\delta^0} \bigoplus_{H \in \mathcal{A}} \dfrac{S}{J(H)}$, where $J(H) = \langle \alpha_H^{\mathbf{m}(H)} \rangle$. That $\mathcal{D}^0 = S^r$ and $\mathcal{D}^1 = \bigoplus_{H \in \mathcal{A}} S/J(H)$ follows from the definition of $\mathcal{D}^\bullet$ and Remark 3.3. To prove that $\mathcal{D}^k = 0$ for $k > 1$, it suffices to show that $D_2(\mathcal{A}_Y, \mathbf{m}_Y) = 0$ for all $Y \in L_2$. We have

$$D_2(\mathcal{A}_Y, \mathbf{m}_Y) = \operatorname{coker}\left(S^r \xrightarrow{\delta_Y^0} \bigoplus_{H \in \mathcal{A}_Y} \frac{S}{J(H)}\right).$$

Since $\mathcal{A}$ is generic, the set $\{\alpha_H : H \in \mathcal{A}_Y\}$ consists of $r(Y)$ linearly independent forms and the coefficient matrix δ^1_Y has full rank. So $D_2(\mathcal{A}_Y, \mathbf{m}_Y) = 0$.

It follows that $H^1(\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})) = \operatorname{coker}(\delta^0)$. Since $|\mathcal{A}| > r$, we see that δ^0 cannot be surjective, so $H^1(\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})) \neq 0$. We show that $H^1(\mathcal{D}^\bullet)$ is only supported at the maximal ideal. To this end, let $P \in \operatorname{spec}(S)$ be a prime of codimension $k \leq r-1$. Write $X(P) = \bigcap\limits_{\substack{H \in \mathcal{A}_X \\ \alpha_H \in P}} H$. Since $\mathcal{A}$ is generic, $\{\alpha_H : \alpha_H \in P\}$ consists of at most k linearly independent forms, so up to a change of coordinates $\mathcal{A}_{X(P)}$ is union of coordinate hyperplanes. By Lemma 3.9, $\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})_P \cong \mathcal{D}^\bullet(\mathcal{A}_{X(P)}, \mathbf{m}_{X(P)})_P$. The chain complex $\mathcal{D}^\bullet(\mathcal{A}_{X(P)}, \mathbf{m}_{X(P)})$ has the form $S^r \xrightarrow{\delta^0_{X(P)}} \bigoplus\limits_{H \in \mathcal{A}_{X(P)}} \dfrac{S}{J(H)}$, and $\delta^0_{X(P)}$ is clearly surjective, so

$$H^1(\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m}))_P \cong H^1(\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})_P) \cong H^1(\mathcal{D}^\bullet(\mathcal{A}_{X(P)}, \mathbf{m}_{X(P)})_P) = 0.$$

It follows that $H^1(\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m}))$ is only supported at the maximal ideal. Since $H^1(\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})) \neq 0$, $\operatorname{pdim}(H^1(\mathcal{D}^\bullet)) = r$ and by Proposition 3.14, $\operatorname{pdim}(D(\mathcal{A}, \mathbf{m})) = r-2$, the maximal projective dimension. $\square$

Remark 3.16. Corollary 3.15 implies that generic arrangements are totally non-free; this was first proved by Yoshinaga [36].

Remark 3.17. Even for simple arrangements, the lower bound given by Corollary 3.15 may be arbitrarily far off from the actual projective dimension. See Remark 6.13.

4. Multi-arrangements and k-formality

In this section we will show that if $(\mathcal{A}, \mathbf{m})$ is a free multi-arrangement then $\mathcal{A}$ is k-formal (in the sense of [12]) for $2 \leq k \leq r-1$, where $r = r(\mathcal{A})$ is the rank of $\mathcal{A}$ (thus generalizing the result of Brandt and Terao [12] to multi-arrangements). Once we have set up the notation, this is an immediate corollary of Theorem 1.1.

We again follow the presentation in [12]. Fix an arrangement $\mathcal{A} = \cup_{H \in \mathcal{A}} V(\alpha_H) \subset V$. Set $E(\mathcal{A}) := \bigoplus_{H \in \mathcal{A}} e_H \mathbb{K}$ and define $\phi : E(\mathcal{A}) \to V^*$ by $\phi(e_H) = \alpha_H$. Put $F(\mathcal{A}) = \ker(\phi)$; this is called the *relation space* of $\mathcal{A}$.

The arrangement $\mathcal{A}$ is 2-*formal* (or just *formal*) if the relation space is generated by relations among three linear forms. Since three linear forms are dependent if and only if they define a codimension two flat, 2-formality is equivalent to surjectivity of the map

$$\pi_2 : \bigoplus_{X \in L_2} F(\mathcal{A}_X) \to F(\mathcal{A}),$$

where π_2 is the sum of natural inclusions $F(\mathcal{A}_X) \hookrightarrow F(\mathcal{A})$ for each $X \in L_2$.

Definition 4.1. Set $R_0 := T(\mathcal{A})^* \subset V^*$, where $T(\mathcal{A}) = \cap_{H \in \mathcal{A}} H$. For $1 \leq k \leq r$, recursively define $R_k(\mathcal{A})$ as the kernel of the map

$$\pi_{k-1} = \pi_{k-1}(\mathcal{A}) := \bigoplus_{X \in L_{k-1}} R_{k-1}(\mathcal{A}_X) \to R_{k-1}(\mathcal{A}),$$

where π_k is the sum of natural inclusions for $0 \leq k \leq r-1$. To simplify notation, set $\mathcal{R}_k = \mathcal{R}_k(\mathcal{A}) = \bigoplus_{X \in L_k} R_k(\mathcal{A}_X)$.

Remark 4.2. After chasing through the definitions one can see that $R_1(\mathcal{A})$ is the kernel of the restriction map $V^* \to T(\mathcal{A})^*$ and $R_2(\mathcal{A}) = F(\mathcal{A})$. See [12] for details.

Definition 4.3. The arrangement $\mathcal{A}$ is

- 2-formal if $\mathcal{A}$ is formal
- k-formal, for $3 \leq k \leq r-1$, if $\mathcal{A}$ is $(k-1)$-formal and the map $\pi_k : \mathcal{R}_k = \bigoplus_{X \in L_k} R_k(\mathcal{A}_X) \to R_k(\mathcal{A})$ is surjective.

In [32], Tohaneanu gives a homological formulation of k-formality as follows. First, notice that there is a natural differential $\delta_k : \mathcal{R}_k \to \mathcal{R}_{k-1}$ (similar to the differential for $\mathcal{D}^\bullet$) defined as the composition $\mathcal{R}_k \to R_k(\mathcal{A}) \xrightarrow{\pi_{k-1}} \mathcal{R}_{k-1}$.

Lemma 4.4. [32, Lemma 2.5] *With the differentials δ_k, $1 \leq k \leq r$, the vector spaces $\mathcal{R}_i$ $(0 \leq k \leq r)$ form a chain complex $\mathcal{R}_\bullet = \mathcal{R}_\bullet(\mathcal{A})$. The arrangement $\mathcal{A}$ is k-formal if and only if $H_i(\mathcal{R}_\bullet) = 0$ for $i = 1, \ldots, k-1$.*

Remark 4.5. If $\mathbf{m} \equiv 1$ (so $(\mathcal{A}, \mathbf{m})$ is a simple arrangement) we will denote $D_k(\mathcal{A}, \mathbf{m})$ (recall Definition 3.1) and $\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})$ (recall Definition 3.5) by $D_k(\mathcal{A})$ and $\mathcal{D}^\bullet(\mathcal{A})$, respectively.

Brandt and Terao show that the vector spaces $R_k(\mathcal{A})$ are dual to the degree zero part of $D_k(\mathcal{A})$.

Proposition 4.6. [12, Proposition 4.10] *For $0 \leq k \leq r$, $D_k(\mathcal{A})_0 \cong R_k(\mathcal{A})^*$, where $R_k(\mathcal{A})^*$ is the $\mathbb{K}$-vector space dual of $R_k(\mathcal{A})$.*

Lemma 4.7. *The modules $D_k(\mathcal{A}, \mathbf{m})$ for $1 \leq k \leq r$ are generated in degree zero. More precisely, we have an isomorphism (as $\mathbb{K}$-vector spaces) $D_k(\mathcal{A}, \mathbf{m})_0 \cong D_k(\mathcal{A})_0$.*

Proof. Both claims are clear for $D_1(\mathcal{A}, \mathbf{m})$ by Remark 3.2. By Definition 3.1, $D_k(\mathcal{A}, \mathbf{m})$ is a quotient of $\bigoplus_{X \in L_k} D_{k-1}(\mathcal{A}_X, \mathbf{m}_X)$. Hence by induction, $D_k(\mathcal{A}, \mathbf{m})$ is also generated in degree zero. Now we have the following commutative diagram:

$$\begin{array}{ccccccc}
D_{k-1}(\mathcal{A}, \mathbf{m})_0 & \xrightarrow{\tau_{k-1}(\mathcal{A})} & \bigoplus_{X \in L_{k-1}} D_{k-1}(\mathcal{A}_X, \mathbf{m}_X)_0 & \longrightarrow & D_k(\mathcal{A}, \mathbf{m})_0 & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & & & \\
D_{k-1}(\mathcal{A})_0 & \xrightarrow{\tau_{k-1}(\mathcal{A})} & \bigoplus_{X \in L_{k-1}} D_{k-1}(\mathcal{A}_X)_0 & \longrightarrow & D_k(\mathcal{A})_0 & \longrightarrow & 0,
\end{array}$$

where the first two vertical maps are isomorphisms by induction. Hence there is also an isomorphism $D_k(\mathcal{A}, \mathbf{m})_0 \cong D_k(\mathcal{A})$. $\square$

Corollary 4.8. *An arrangement $\mathcal{A}$ is k-formal if and only if $H^i(\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})_0) = 0$ for $i = 1, \ldots, k-1$.*

Proof. Immediate from Lemma 4.7, Lemma 4.4, and Proposition 4.6. $\square$

Definition 4.9. An arrangement $\mathcal{A}$ is *totally formal* if $\mathcal{A}_X$ is k-formal for $2 \leq k \leq r(X)$ for all $X \in L(\mathcal{A})$.

For example, a rank three arrangement is totally formal if and only if it is formal. See Remark 5.10 for further examples of totally formal arrangements.

Corollary 4.10. *If $(\mathcal{A}, \mathbf{m})$ is free then $\mathcal{A}$ is totally formal.*

Proof. Suppose to the contrary that $\mathcal{A}_X$ is not k-formal for some $X \in L$ and $2 \leq k \leq r(X) - 1$. Then, by Corollary 4.8, $H^i(\mathcal{D}_0^\bullet(\mathcal{A}_X, \mathbf{m}_X)) \neq 0$ for some $1 \leq i \leq k-1$. Hence by Theorem 1.1, $D(\mathcal{A}_X, \mathbf{m}_X)$ is not free, whence $D(\mathcal{A}, \mathbf{m})$ is not free by Proposition 2.4. □

Remark 4.11. We will see in Proposition 6.2 that there are totally formal arrangements which nevertheless are totally non-free. See also Example 6.5.

Remark 4.12. The ranks of the vector spaces appearing in $\mathcal{R}_\bullet$ are not combinatorial in general (see Example 6.5), however if $\mathcal{A}$ is totally formal then these ranks are determined by $L(\mathcal{A})$. We can see this by inductively reading off the rank of $R_k(\mathcal{A}_X)$ ($X \in L_k$) from the Euler characteristic of $\mathcal{R}_\bullet(\mathcal{A}_X)$; since $\mathcal{A}$ is totally formal the Euler characteristic of $\mathcal{R}_\bullet(\mathcal{A}_X)$ is zero by Lemma 4.4. This yields a number of combinatorial obstructions to freeness which can be read off $L(\mathcal{A})$ (see for instance [12, Corollary 4.16]). By Corollary 4.10, if any of these combinatorial obstructions are satisfied, the arrangement is totally non-free.

In the following corollary, we call a hyperplane $H \in \mathcal{A}$ *generic* if, for all $X \in L_2$ so that $H < X$ in L, there is a unique hyperplane $H' \neq H$ so that $H' < X$. Moreover, we say H is a *separator* of $\mathcal{A}$ if $r(\mathcal{A} - H) < r(\mathcal{A})$. Part of the following result may be found in [12, Proposition 3.9]; we provide a proof for completeness.

Corollary 4.13. *Suppose $\mathcal{A}$ is an arrangement of rank ≥ 2. If $\mathcal{A}$ has a generic hyperplane which is not a separator, then $\mathcal{A}$ is not formal. In particular, $\mathcal{A}$ is totally non-free.*

Proof. Let $H \in \mathcal{A}$ be the generic hyperplane which is not a separator, and write v_H for the corresponding row of δ_S^0. The condition that H is not a separator means that we can find $r = r(\mathcal{A})$ linearly independent rows $v_1, \ldots, v_r$ of δ_S^0 where $v_i \neq v_H$ for $i = 1, \ldots, r$. Hence there is a relation $\sum_{i=1}^r c_i v_i + c_H v_H = 0$ (for constants $c_1, \ldots, c_H$). Since $r \geq 2$ and H is generic, there is no way to write this relation as a linear combination of relations among three hyperplanes (since v_H is not in the support of any such relation). So $\mathcal{A}$ is not formal. The final conclusion follows from Corollary 4.10. □

5. Computing the chain complex

In this section we work out concrete presentations for the modules appearing in $\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})$ and illustrate the constructions via examples, with the goal of studying freeness and projective dimension of $D(\mathcal{A}, \mathbf{m})$. The following definition, which constructs $\mathcal{D}^\bullet$ as the cokernel of a map of chain complexes, is analogous to the setup of the Billera-Schenck-Stillman chain complex used in algebraic spline theory [10, 27]. Since there are many details, the reader may find it easiest to read the following constructions while following along with Examples 5.7 and 5.8.

Definition 5.1. For a multi-arrangement $(\mathcal{A}, \mathbf{m})$, set $S_k(\mathcal{A}_X) = D_k(\mathcal{A}_X, \mathbf{m}_X)_0 \otimes_{\mathbb{K}} S$, the degree zero part of $D_k(\mathcal{A}_X, \mathbf{m}_X)$ tensored with S, and set $\mathcal{S}^\bullet(\mathcal{A}) := \mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})_0 \otimes_{\mathbb{K}} S$, so $\mathcal{S}^k = \bigoplus_{X \in L_k} S_k(\mathcal{A}_X)$. These are independent of the choice of multiplicities by Lemma 4.7.

For $Y \in L$, write $\phi_k^S(Y), \tau_k^S$ for the maps $\phi_k^S(Y) : S_k(\mathcal{A}) \to S_k(\mathcal{A}_Y), \tau_k^S : S_k(\mathcal{A}) \to \bigoplus_{X \in L_k} S_k(\mathcal{A}_X)$ which are obtained from the maps $\phi_k(Y) : D_k(\mathcal{A}, \mathbf{m}) \to$

$$\begin{tikzcd}
\mathcal{J}^\bullet(\mathcal{A},\mathbf{m})\cdots \arrow[r] & \bigoplus_{X\in L_{k-1}} J_{k-1}(\mathcal{A}_X,\mathbf{m}_X) \arrow[r, "\delta_J^{k-1}"] \arrow[d] & \bigoplus_{Y\in L_k} J_k(\mathcal{A}_Y,\mathbf{m}) \arrow[r, "\delta_J^k"] \arrow[d] & \cdots \\
\mathcal{S}^\bullet(\mathcal{A})\cdots \arrow[r] & \bigoplus_{X\in L_{k-1}} S_{k-1}(\mathcal{A}_X) \arrow[r, "\delta_S^{k-1}"] \arrow[d] & \bigoplus_{Y\in L_k} S_k(\mathcal{A}_Y) \arrow[r, "\delta_S^k"] \arrow[d] & \cdots \\
\mathcal{D}^\bullet(\mathcal{A},\mathbf{m})\cdots \arrow[r] & \bigoplus_{X\in L_{k-1}} D_{k-1}(\mathcal{A}_X,\mathbf{m}_X) \arrow[r, "\delta^{k-1}"] & \bigoplus_{Y\in L_k} D_k(\mathcal{A}_Y,\mathbf{m}) \arrow[r, "\delta^k"] & \cdots
\end{tikzcd}$$

FIGURE 5. Short exact sequence of complexes from Definition 5.1

$D_k(\mathcal{A}_Y,\mathbf{m}), \tau_k : D_k(\mathcal{A},\mathbf{m}) \to \bigoplus_{X\in L_k} D(\mathcal{A}_X,\mathbf{m}_X)$ (see Definition 3.1) by restricting to degree zero and then tensoring with S. Likewise write δ_S^i for the differential of $\mathcal{S}^\bullet$.

Since each of the modules $D_k(\mathcal{A},\mathbf{m})$ is generated in degree zero by Lemma 4.7, there is a natural surjective map $S_k(\mathcal{A}_X) \to D_k(\mathcal{A}_X,\mathbf{m}_X)$ for every $\mathbf{m}$ and $X \in L_k$. Hence there is a surjective map of complexes $\mathcal{S}^\bullet(\mathcal{A}) \to \mathcal{D}^\bullet(\mathcal{A},\mathbf{m})$ for any multiplicity $\mathbf{m}$.

For each surjection $S_k(\mathcal{A}_X) \to D_k(\mathcal{A}_X,\mathbf{m}_X)$, write $J_k(\mathcal{A}_X,\mathbf{m}_X)$ for the kernel of this surjection, and write $\mathcal{J}^\bullet(\mathcal{A},\mathbf{m})$ for the kernel of the surjection $\mathcal{S}^\bullet(\mathcal{A}) \to \mathcal{D}^\bullet(\mathcal{A},\mathbf{m})$, so $\mathcal{J}^k(\mathcal{A},\mathbf{m}) = \bigoplus_{X\in L_k} J_k(\mathcal{A}_X,\mathbf{m}_X)$. Denote by $\phi_i^J(Y), \tau_i^J$, and δ_J^i the maps obtained from restricting $\phi_i^S(Y), \tau_i^S$, and δ_S^i. See figure 5 which shows the short exact sequence of complexes $0 \to \mathcal{J}^\bullet \to \mathcal{S}^\bullet \to \mathcal{D}^\bullet \to 0$.

Remark 5.2. By Corollary 4.8, $\mathcal{A}$ is k-formal if and only if $H^i(\mathcal{S}^\bullet(\mathcal{A})) = 0$ for $1 \le i \le k-1$. Furthermore $\mathcal{A}$ is essential if and only if $H^0(\mathcal{S}^\bullet(\mathcal{A})) = 0$.

Remark 5.3. The short exact sequence $0 \to \mathcal{J}^\bullet \to \mathcal{S}^\bullet \to \mathcal{D}^\bullet \to 0$ gives rise to a long exact sequence starting as

$$0 \to H^0(\mathcal{S}^\bullet) \to H^0(\mathcal{D}^\bullet) \cong D(\mathcal{A},\mathbf{m}) \xrightarrow{\psi} H^1(\mathcal{J}^\bullet) \to H^1(\mathcal{S}^\bullet) \to \cdots,$$

where ψ is defined on $\theta \in D(\mathcal{A},\mathbf{m})$ as $\psi(\theta) = \sum_{H\in L_1} \theta(\alpha_H) \in \oplus_{H\in L_1} J(H)$. The map ψ is an isomorphism if (and only if) $\mathcal{A}$ is essential and formal.

Remark 5.4. If $\mathcal{A}$ is essential and k-formal for all $k \ge 2$, then the long exact sequence from Remark 5.3 breaks into isomorphisms $H^i(\mathcal{D}^\bullet(\mathcal{A},\mathbf{m})) \cong H^i(\mathcal{J}^\bullet(\mathcal{A},\mathbf{m}))$ for $i \ge 0$ (by Remark 5.2). In particular, if we wish to determine free multiplicities on an arrangement, we may assume by Corollary 4.10 that $\mathcal{A}$ is k-formal for all $k \ge 2$, hence the isomorphism $H^i(\mathcal{D}^\bullet(\mathcal{A},\mathbf{m})) \cong H^i(\mathcal{J}^\bullet(\mathcal{A},\mathbf{m}))$ holds for $i \ge 0$.

Lemma 5.5. *Let $(\mathcal{A},\mathbf{m})$ be a multi-arrangement. If $H \in L_1$, then set $J(H) = J_1(\mathcal{A}_H,\mathbf{m}(H)) = \langle \alpha_H^{\mathbf{m}(H)} \rangle$. If $X \in L_k$ where $k > 1$, then the module $J_k(\mathcal{A}_X,\mathbf{m}_X)$*

satisfies

$$\begin{aligned} J_k(\mathcal{A}_X, \mathbf{m}_X) &= \delta_S^{k-1}\left(\bigoplus_{\substack{Y \in L_{k-1} \\ X<Y}} J_{k-1}(\mathcal{A}_Y, \mathbf{m}_Y)\right) \\ &= \sum_{\substack{Y \in L_{k-1} \\ X<Y}} \phi_{k-1}^S(X)(J_{k-1}(\mathcal{A}_Y, \mathbf{m}_Y)) \end{aligned}$$

with $\delta_S^{k-1} : \mathcal{S}^{k-1} \to \mathcal{S}^k$ *and* $\phi_S^k(X) : S_k(\mathcal{A}_Y) \to S_k(\mathcal{A}_X)$ *the maps from Definition 5.1.*

Proof. For simplicity we take $\mathcal{A}_X = \mathcal{A}$, so $\mathcal{A}$ has rank k and $X = \cap_{H \in \mathcal{A}} H$. The tail end of the short exact sequence of complexes $0 \to \mathcal{J}^\bullet \to \mathcal{S}^\bullet \to \mathcal{D}^\bullet \to 0$ is shown below.

$$\begin{array}{ccccc}
\mathcal{J}^{k-2} & \xrightarrow{\delta_J^{k-2}} & \mathcal{J}^{k-1} = \bigoplus_{Y \in L_{k-1}} J_{k-1}(\mathcal{A}_Y, \mathbf{m}_Y) & \xrightarrow{\delta_J^{k-1}} & \mathcal{J}^k = J_k(\mathcal{A}, \mathbf{m}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{S}^{k-2} & \xrightarrow{\delta_S^{k-2}} & \mathcal{S}^{k-1} = \bigoplus_{Y \in L_{k-1}} S_{k-1}(\mathcal{A}_Y) & \xrightarrow{\delta_S^{k-1}} & \mathcal{S}^k = S_k(\mathcal{A}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{D}^{k-2} & \xrightarrow{\delta^{k-2}} & \mathcal{D}^{k-1} = \bigoplus_{Y \in L_{k-1}} D_{k-1}(\mathcal{A}, \mathbf{m}) & \xrightarrow{\delta^{k-1}} & \mathcal{D}^k = D_k(\mathcal{A}, \mathbf{m})
\end{array}$$

The differentials δ_S^{k-2} and δ^{k-2} factor through $S_{k-1}(\mathcal{A})$ and $D_{k-1}(\mathcal{A}, \mathbf{m})$, respectively, by Definition 3.5. It follows that $H^{k-1}(\mathcal{S}^\bullet) = H^{k-1}(\mathcal{D}^\bullet) = H^k(\mathcal{S}^\bullet) = H^k(\mathcal{D}^\bullet) = 0$ by Definition 3.1. Hence the long exact sequence in cohomology yields that $H^k(\mathcal{J}^\bullet) = 0$, in other words δ_J^{k-1} is surjective. The first equality follows from commutativity of the diagram. By definition, $\delta_J^k = \tau_k^J = \sum_{Y \in L_{k-1}} \phi_{k-1}^J(X)$. Since $\phi_{k-1}^J(X)$ is the restriction of $\phi_{k-1}^S(X)$, this proves the second equality. □

From Lemma 5.5, we see that in order to explicitly determine the complexes $\mathcal{J}^\bullet$ and $\mathcal{D}^\bullet$, it suffices to determining the maps $\phi_k^S(Y)$ for $Y \in L_k$, or equivalently to determine the differential δ_S^k of the complex $\mathcal{S}^\bullet$. In § 4, we saw that $\mathcal{S}^\bullet \cong (\mathcal{R}_\bullet^*) \otimes_{\mathbb{K}} S$, so the differential δ_S^k is just the transpose of the differential δ_k in the complex $\mathcal{R}_\bullet$. By examining these matrices as they appear in [12] and [32], we obtain the following recipe for constructing δ_S^k.

Lemma 5.6. *A matrix for δ_S^k may be inductively defined as follows. The matrix for δ_S^0 is the coefficient matrix for $\mathcal{A}$, whose rows give coefficients of the linear forms defining $\mathcal{A}$. Inductively, δ_S^k may be represented by a matrix whose rows are naturally grouped according to flats $X \in L_k$. A row corresponding to $X \in L_k$ encodes a relation among rows of δ_S^{k-1} which correspond to flats $Y \in L_{k-1}$ so that $Y < X$; the set of all rows corresponding to $X \in L_k$ is a choice of basis for all relations among the rows of δ_S^{k-1} corresponding to flats $Y \in L_k$ so that $Y < X$.*

Example 5.7 (Points in $\mathbb{P}^1$)**.** Consider the arrangement $\mathcal{A}$ of $k+2$ points in $\mathbb{P}^1$, corresponding to the product $xy(x-a_1y)\dots(x-a_ky)$. Let $H_x = V(x), H_y = V(y)$, and $H_i = V(x-a_iy)$ for $i = 1,\dots,k$. By Lemma 5.6, the complex $\mathcal{S}^\bullet$ is

$$0 \to S^2 \xrightarrow{\delta^0_S} S^{k+2} \xrightarrow{\delta^1_S} S^k \to 0,$$

where

$$\delta^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -a_1 \\ \vdots & \vdots \\ 1 & -a_k \end{bmatrix} \qquad \text{and} \qquad \delta^1 = \begin{bmatrix} -1 & a_1 & 1 & 0 & \cdots & 0 \\ -1 & a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ -1 & a_k & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Notice that $S_2(\mathcal{A}) \cong S^k$. Write m_x, m_y for $\mathbf{m}(H_x), \mathbf{m}(H_y)$, respectively, and m_i for $\mathbf{m}(H_i)$, $i = 1,\dots,k$. By Lemma 5.5, $J_2(\mathcal{A},\mathbf{m}) = \mathcal{J}^2(\mathcal{A},\mathbf{m})$ is generated by the columns of the matrix

$$M = \begin{bmatrix} -x^{m_x} & a_1y^{m_y} & (x-a_1y)^{m_1} & 0 & \cdots & 0 \\ -x^{m_x} & a_2y^{m_y} & 0 & (x-a_2y)^{m_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ -x^{m_x} & a_ky^{m_y} & 0 & 0 & \cdots & (x-a_ky)^{m_k} \end{bmatrix},$$

so $D^2(\mathcal{A},\mathbf{m}) \cong \operatorname{coker}(M)$. Notice that M is a matrix for δ^1_J with the natural choice of basis for $\bigoplus_{H\in L_1} J(H) \cong \bigoplus_{H\in L_1} S(-\mathbf{m}(H))$. Hence, by Remark 5.4, we may identify $D(\mathcal{A},\mathbf{m})$ with $H^1(\mathcal{J}^\bullet,\mathbf{m})$, which is exactly the syzygies on the columns of M (it is also straightforward to see this from the definition of $D(\mathcal{A},\mathbf{m})$). In particular, if $k = 1$ so $\mathcal{A}$ is the A_2 braid arrangement, then $D(A_2,\mathbf{m})$ may be identified with the syzygies on the forms x^{m_x}, y^{m_y}, and $(x-a_1y)^{m_1}$. This provides an alternative way to identify the generators and exponents of $(A_2,\mathbf{m})$, which were originally found in [33] (see [18],[13, Example 3.6,Lemma 4.5] for more details).

For an arrangement defined by the vanishing of forms $\alpha_1,\dots,\alpha_n$, we will write H_i for $V(\alpha_i)$ and denote the flat $H_{i_1}\cap\cdots\cap H_{i_k}$ by the list of indices $i_1\cdots i_k$. Furthermore, we will denote by L_2^{trip} the set of rank two flats which are the intersection of at least three hyperplanes.

Example 5.8 (X_3 arrangement)**.** Consider the arrangement $\mathcal{A}_t$ defined by the vanishing of the six linear forms

$$\begin{aligned} \alpha_1 &= x & \alpha_4 &= x - ty \\ \alpha_2 &= y & \alpha_5 &= x + z \\ \alpha_3 &= z & \alpha_6 &= y + z. \end{aligned}$$

The intersection lattice of $\mathcal{A}_t$ is constant as long as $t \neq 0, 1$, with six double points and three triple points $L_2^{\text{trip}} = \{124, 135, 236\}$. Lemma 5.6 yields

$$\mathcal{S}^\bullet = 0 \to S^3 \xrightarrow{\delta^0_S} S^6 \xrightarrow{\delta^1_S} S^3 \to 0,$$

where

$$\delta_S^0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -t & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \end{array} \qquad \delta_S^1 = \begin{array}{c} \\ 124 \\ 135 \\ 236 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & -t & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \end{pmatrix} \end{array}.$$

This complex is always exact, hence $\mathcal{A}_t$ is always formal for $t \neq 0, 1$ by Corollary 4.8. By Remark 5.4, $H^i(\mathcal{D}^\bullet) \cong H^i(\mathcal{J}^\bullet)$. By Theorem 1.1, we may check freeness of $D(\mathcal{A}_t, \mathbf{m})$ by determining vanishing of $H^1(\mathcal{J}^\bullet)$.

Now we consider the complex $\mathcal{J}^\bullet$. Write $J(i)$ for $J_1((\mathcal{A}_t)_{H_i}, m_i) = \langle \alpha_i^{m_i} \rangle$. If $ijk \in L_2^{\text{trip}}$, we write $J(ijk)$ for the ideal $J(i)+J(j)+J(k)$, where $ijk \in L_2^{\text{trip}}$. Then, by Lemma 5.5, $J_2(124, \mathbf{m}) = J(1) - tJ(3) - J(4) = J(1) + J(3) + J(4) = J(134)$. The same holds for any triple point, so $J_2(ijk, \mathbf{m}) = J(ijk)$ for every $ijk \in L_2^{\text{trip}}$. So $\mathcal{J}^2 = \oplus_{ijk \in L_2^{\text{trip}}} J(ijk)$ and

$$\mathcal{J}^\bullet = 0 \to \bigoplus_{i=1}^{6} J(i) \xrightarrow{\delta_J^1} \bigoplus_{ijk \in L_2^{\text{trip}}} J(ijk),$$

where δ_J^1 is the restriction of δ_S^1. A presentation for $H^2(\mathcal{J}^\bullet)$ is worked out in [14] and is used to prove that $(\mathcal{A}_t, \mathbf{m})$ is free if and only if the defining equation has the form $\mathcal{Q}(\mathcal{A}, \mathbf{m}) = x^n y^n z^n (x - ty)(x + z)(y + z)$, where $t^n = 1$. We generalize this result in Theorem 6.10.

5.1. **Graphic arrangements.** Let G be a simple graph (no loops or multiple edges) on ℓ vertices $\{v_1, \dots, v_\ell\}$ with edge set $E(G)$, $S = \mathbb{K}[x_1, \dots, x_\ell]$ (with x_i corresponding to v_i), and set $H_{ij} = V(x_i - x_j)$. The *graphic arrangement* associated to G is the arrangement $\mathcal{A}_G = \cup_{\{v_i, v_j\} \in E(G)} H_{ij}$; $\mathcal{A}_G$ is a sub-arrangement of the $A_{\ell-1}$. A multiplicity $\mathbf{m}$ on $\mathcal{A}_G$ is determined by the values $m_{ij} = \mathbf{m}(H_{ij})$ corresponding to edges $\{v_i, v_j\} \in E(G)$.

Recall that the *clique complex* (or *flag complex*) of a graph G is the simplicial complex $\Delta = \Delta(G)$ with an i-simplex for every complete graph on $(i-1)$ vertices.

Lemma 5.9. *The chain complex $\mathcal{S}^\bullet(\mathcal{A}_G)$ may be identified with the simplicial cochain complex of $\Delta(G)$ with coefficients in S. Hence $\mathcal{A}_G$ is k-formal if and only if $H^i(\Delta(G); S) = 0$ for $1 \le i \le k-1$.*

Proof. By [32, Lemma 3.1], $\mathcal{R}_\bullet(\mathcal{A}_G)$ may be identified with the simplicial chain complex of $\Delta(G)$ with coefficients in $\mathbb{K}$. Now use the isomorphism $\mathcal{S}^\bullet \cong (\mathcal{R}_\bullet)^* \otimes_{\mathbb{K}} S$. □

Remark 5.10. Using Lemma 5.9 we may easily see how the notions of k-formal for various k are distinct; this was part of the intent of [32]. This lemma also makes it clear that the condition that $\mathcal{A}_G$ is k-formal for $2 \le k \le r-1$ is distinct from the condition of being totally formal. A graphic arrangement $\mathcal{A}_G$ is k-formal for $2 \le k \le r-1$ if and only if its clique complex $\Delta(G)$ is contractible. On the other hand, $\mathcal{A}_G$ is totally formal if and only if G is chordal; a much stronger condition which coincides with both freeness and supersolvability of $\mathcal{A}_G$ [30].

If $\sigma \in \Delta(G)_k$ is a complete graph on the $(k+1)$ vertices $\{v_{i_0}, \ldots, v_{i_k}\}$ (where $k \geq 1$), then write $J(\sigma)$ for the ideal generated by the forms $\{(x_{i_s} - x_{i_t})^{m_{i_s i_t}} : 0 \leq s < t \leq k\}$. If $\sigma = \{v_i\}$ is a single vertex, then we take $J(\sigma) = 0$.

Proposition 5.11. *If G is a simple graph, then $\mathcal{D}^\bullet(\mathcal{A}_G, \mathbf{m})$ has modules*

$$\mathcal{D}^i \cong \bigoplus_{\sigma \in \Delta(G)_i} S/J(\sigma)$$

and differentials δ^i induced from the simplicial co-chain complex with coefficients in S, which may be identified with $\mathcal{S}^\bullet(\mathcal{A}_G)$.

Proof. Use the identification of the differentials δ^i in Lemma 5.9 as the simplicial co-chain differential for $\Delta(G)$ and the construction of $J_k((\mathcal{A}_G)_X, \mathbf{m}_X)$ from Lemma 5.5. $\square$

Remark 5.12. The chain complex in Proposition 5.11 was introduced in [15] by analogy with a natural class of chain complexes in the context of multivariate spline theory [10, 27]. Applying Theorem 1.1 yields the homological characterization of freeness obtained in [15, Corollary 5.6].

Remark 5.13. The first non-trivial classification of free multiplicities on a graphic arrangement admitting both free and non-free multiplicities was completed in [2]. Building on work of Abe, Nuida, and Numata [5], the classification of free multiplicities on the A_3 braid arrangement has been completed in [13]. The key is a detailed analysis of $H^2(\mathcal{D}^\bullet(A_3, \mathbf{m}))$, where $\mathcal{D}^\bullet$ is the complex described in Corollary 5.11.

6. TF_2 ARRANGEMENTS

In this section we introduce a subset of the totally formal arrangements which we shall call TF_k arrangements. These are totally formal arrangements which additionally satisfy that $\mathcal{S}^i(\mathcal{A}) = 0$ for $i > k$. For instance, every totally formal arrangement is TF_k for $k \geq r(\mathcal{A})$. A graphic arrangement $\mathcal{A}_G$ is TF_k if and only if G is chordal (see Remark 5.10) and $\dim(\Delta(G)) \leq k$. By Theorem 1.1 and Remark 5.4, freeness of TF_k arrangements is determined by the vanishing of $H^i(\mathcal{J}^\bullet)$ for $2 \leq i \leq k$. In the rest of this section we will assume that $\mathcal{A}$ is a TF_2 arrangement of rank at least three.

6.1. Free TF_2 arrangements.

Recall that an arrangement $\mathcal{A}$ is *supersolvable* if there is a filtration $\mathcal{A}_1 \subset \cdots \subset \mathcal{A}_r = \mathcal{A}$ satisfying the following rank property (RP) and intersection property (IP):

- (RP) $r(\mathcal{A}_i) = i$ for $i = 1, \ldots, r(\mathcal{A})$.
- (IP) For any $H, H' \in \mathcal{A}_i$ there exists some $H'' \in \mathcal{A}_{i-1}$ so that $H \cap H' \subset H''$.

Proposition 6.1. *Let $\mathcal{A}$ be an irreducible TF_2 arrangement of rank $r = r(\mathcal{A})$. Then*

- $|\mathcal{A}| = r - \#L_2^{trip} + \sum_{X \in L_2^{trip}} (|\mathcal{A}_X| - 1)$
- $|\mathcal{A}| \leq 1 + \sum_{X \in L_2^{trip}} (|\mathcal{A}_X| - 1)$
- $\#L_2^{trip} \geq r - 1$

Furthermore, the following are equivalent.

1. $\mathcal{A}$ *is free*
2. $|\mathcal{A}| = 1 + \sum_{X \in L_2^{trip}} (|\mathcal{A}_X| - 1)$

(3) $\#L_2^{trip} = r - 1$
(4) *$\mathcal{A}$ is supersolvable*

In particular, if $\mathcal{A}$ is TF_2, its freeness may be determined from $L(\mathcal{A})$.

Proof. The first three bullet points are computed from the Euler characteristic of $\mathcal{S}^\bullet(\mathcal{A})$ and $\mathcal{J}^\bullet(\mathcal{A})_1$ as follows. Since $\mathcal{A}$ is TF_2, $\mathcal{S}^\bullet(\mathcal{A})$ is a short exact sequence of the form:

$$0 \to S^\ell = S^r \to S^{|\mathcal{A}|} \to \bigoplus_{X \in L_2^{\text{trip}}} S^{|\mathcal{A}_X|-2} \to 0,$$

so the alternating sum of the ranks yields $|\mathcal{A}| = r + \sum_{X \in L_2^{\text{trip}}}(|\mathcal{A}_X| - 2) = r - \#L_2^{\text{trip}} + \sum_{X \in L_2^{\text{trip}}}(|\mathcal{A}_X| - 1)$. For the second bullet point, $\mathcal{J}^\bullet(\mathcal{A})$ has the form

$$0 \to \bigoplus_{H \in \mathcal{A}} J(H) \xrightarrow{\delta_J^1} \bigoplus_{X \in L_2^{\text{trip}}} J_2(\mathcal{A}_X) \to 0.$$

Since $\ker(\delta_J^1) = D(\mathcal{A})$ and we assumed $\mathcal{A}$ is irreducible, $\ker(\delta_J^1)_1$ is one dimensional, spanned by the Euler derivation. We may easily compute $\dim J_2(\mathcal{A}_X)_1 = |\mathcal{A}_X| - 1$ for $X \in L_2^{\text{trip}}$, hence

$$\dim H^2(\mathcal{J}^\bullet)_1 = \sum_{X \in L_2^{\text{trip}}} (|\mathcal{A}_X| - 1) - |\mathcal{A}| + 1$$

by computing the Euler characteristic of $\mathcal{J}_1^\bullet$. This must be non-negative, yielding $|\mathcal{A}| \leq 1 + \sum\limits_{X \in L_2^{\text{trip}}} (|\mathcal{A}_X| - 1)$. The third bullet point follows from putting the first two bullet points together.

Now we prove the equivalent conditions for freeness. The implication (4) $\implies$ (1) is a well known fact. Since supersolvability is determined from $L(\mathcal{A})$, the final statement is immediate from (4). We first prove (1) $\iff$ (2). From Theorem 1.1 and Remark 5.4, $\mathcal{A}$ is free if and only if $H^2(\mathcal{J}^\bullet) = 0$. From the explicit description in Example 5.7, we see that $J_2(\mathcal{A}_X)$ is generated in degree one for every $X \in L_2^{\text{trip}}$, as is $J(H) \cong \langle \alpha_H \rangle$ for every $H \in \mathcal{A}$. So $H^2(\mathcal{J}^\bullet)$ must also be generated in degree one since it is a quotient of $\sum_{X \in L_2^{\text{trip}}} J_2(\mathcal{A}_X)$. From our above computation,

$$\dim H^2(\mathcal{J}^\bullet)_1 = \sum_{X \in L_2^{\text{trip}}} (|\mathcal{A}_X| - 1) - |\mathcal{A}| + 1,$$

hence $\mathcal{A}$ is free if and only if this expression vanishes, i.e. $|\mathcal{A}| = 1 + \sum\limits_{X \in L_2^{\text{trip}}} (|\mathcal{A}_X| - 1)$. (3) follows immediately from (2) using the expression $|\mathcal{A}| = r - \#L_2^{\text{trip}} + \sum_{X \in L_2^{\text{trip}}}(|\mathcal{A}_X| - 1)$ already proved. Finally, we show (3) $\implies$ (4). First, for any $X, X' \in L_2^{\text{trip}}$, we prove there is a sequence $X = X_1, H_1, X_2, \ldots, H_{k-1}, X_k = X'$ satisfying

(1) $H_i \in \mathcal{A}$ for $i = 1, \ldots, k-1$
(2) $X_i \in L_2^{\text{trip}}$ for $i = 1, \ldots, k$.
(3) $H_i < X_i$ and $H_{i+1} < X_{i+1}$ in $L(\mathcal{A})$ for $i = 1, \ldots, k-1$.

To show this, let $H_1, H_2 \in \mathcal{A}_X$ and $H_1', H_2' \in \mathcal{A}_{X'}$ with corresponding linear forms $\alpha_1, \alpha_2, \alpha_1', \alpha_2'$. Complete $\alpha_1, \alpha_2, \alpha_1'$ to a basis B of V^* using defining forms of $\mathcal{A}$ (this is possible because $\mathcal{A}$ is essential). Adding α_2' to B, we see there is a relation

$$\begin{tikzcd}
 & \bigoplus_{H\in\mathcal{A}} J(H) \arrow[r, "\cong"] \arrow[d, "\iota"] & \bigoplus_{H\in\mathcal{A}} J(H) \arrow[d, "\delta^1_J"] \\
\bigoplus_{X\in L_2^{\mathrm{trip}}} D(\mathcal{A}_X,\mathbf{m}_X) \arrow[r, "\sum\psi_X"] & \bigoplus_{X\in L_2^{\mathrm{trip}}}\left[\bigoplus_{H<X} J(H)\right] \arrow[r, "\sum(\delta^1_J)_X"] & \bigoplus_{X\in L_2^{\mathrm{trip}}} J_2(\mathcal{A}_X,\mathbf{m}_X)
\end{tikzcd}$$

FIGURE 6. Diagram for Proposition 6.2

of length $r+1$ among the forms $B \cup \{\alpha_2'\}$. Since $\mathcal{A}$ is formal, this relation can be expressed as a linear combination of relations of length three. We then read off the sequence $X = X_1, H_1, \ldots, X_k = X'$ from this linear combination of relations of length three.

Now we construct a filtration $\mathcal{F} = \mathcal{F}(\mathcal{A}) = \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}_r = \mathcal{A}$ of $\mathcal{A}$. Let $\mathcal{A}_1 = H$ for any $H \in \mathcal{A}$, and $\mathcal{A}_2 = \mathcal{A}_{X_1}$ for some $X_1 \in L_2^{\mathrm{trip}}$ so that $H \in \mathcal{A}_{X_1}$ (by Corollary 4.13, every $H \in \mathcal{A}$ passes through some $X \in L_2^{\mathrm{trip}}$). Build $\mathcal{A}_{i+1}$ from $\mathcal{A}_i$ for $2 \le i \le r$ inductively as follows. By our above claim, there exists $X_i \in L_2^{\mathrm{trip}}$ so that $\mathcal{A}_i \cap \mathcal{A}_{X_i} \neq \emptyset$. Then set $\mathcal{A}_{i+1} = \mathcal{A}_i \cup \mathcal{A}_{X_i}$. This process finishes with $\mathcal{A}_{(r-1)+1} = \mathcal{A}_r$, when we have exhausted L_2^{trip}. Notice that $\mathcal{F}$ satisfies the intersection property (IP) by construction. Moreover, $r(\mathcal{A}_i) \le r(\mathcal{A}_{i-1}) + 1$, hence since the filtration has length r with $\mathcal{A}_r = \mathcal{A}$, we must have $r(\mathcal{A}_i) = i$. Hence $\mathcal{F}(\mathcal{A})$ is a supersolvable filtration. $\square$

6.2. **Presentation for $H^2(\mathcal{J}^\bullet)$.** Assuming $\mathcal{A}$ is a TF_2 arrangement, we now obtain an explicit presentation for $H^2(\mathcal{J}^\bullet(\mathcal{A},\mathbf{m}))$. Consider the diagram in Figure 6, where the chain complex $\mathcal{J}^\bullet$ appears on the right hand side ($\mathcal{J}^\bullet$ has only two terms since $\mathcal{A}$ is TF_2). For book-keeping purposes we use the formal symbols $[H]$ and $[X,H]$ (or $[\alpha_H], [X,\alpha_H]$), of degree $\mathbf{m}(H)$, to denote the generators $\alpha_H^{\mathbf{m}(H)}$ of the summands $J(H) = \langle \alpha_H^{\mathbf{m}(H)} \rangle$ which appear in $\bigoplus_{H\in\mathcal{A}} J(H)$ and $\bigoplus_{X\in L_2^{\mathrm{trip}}} \bigoplus_{H<X} J(H)$, respectively. With this notation, the map $\psi_X : D(\mathcal{A}_X,\mathbf{m}_X) \to \bigoplus_{H<X} J(X)$ in Figure 6 is the map $\psi_X(\theta) = \sum_H \dfrac{\theta(\alpha_H)}{\alpha_H^{\mathbf{m}(H)}}[X,H]$ and $\iota : \bigoplus J(H_i) \to \bigoplus_{X\in L_2^{\mathrm{trip}}} \bigoplus_{X<H_i} J(H_i)$ is the natural inclusion defined by $\iota([H]) = \sum_{X\in L_2^{\mathrm{trip}}} \sum_{X<H} [X,H]$ and extended linearly. The main thing to check for commutativity is that $(\sum(\delta_J^1)_X) \circ \iota = \delta_J^1$, which follows from the definition.

Proposition 6.2. *Suppose $\mathcal{A}$ is an irreducible TF_2 arrangement of rank at least three. Then*

$$H^2(\mathcal{J}^\bullet) \cong \mathit{coker}\left(\bigoplus_{X\in L_2^{\mathit{trip}}} D(\mathcal{A}_X,\mathbf{m}_X) \xrightarrow{\sum \overline{\psi_X}} \mathit{coker}(\iota) \cong S^\kappa \right),$$

where $\kappa = (\sum_{X\in L_2^{\mathit{trip}}} |\mathcal{A}_X|) - |\mathcal{A}|$. Moreover,

1. *$(\mathcal{A},\mathbf{m})$ is free if and only if $\sum \overline{\psi_X}$ is surjective.*
2. *$\kappa > 0$, i.e. $|\mathcal{A}| < \sum_{X\in L_2^{\mathit{trip}}} |\mathcal{A}_X|$.*

(3) *If $|\mathcal{A}| < \sum_{X \in L_2^{trip}}(|\mathcal{A}_X| - 1)$ or equivalently $r < \#L_2^{trip}$ then $\mathcal{A}$ is totally non-free. Furthermore in this case every $\mathcal{A}' \in \mathcal{M}(L(\mathcal{A}))$ is totally non-free.*

Remark 6.3. The presentation in Proposition 6.2 is similar in spirit to a presentation derived in [27, Lemma 3.8] for a homology module which governs freeness of bivariate splines on triangulations.

Proof. Since the commutative diagram in Figure 6 has exact rows, the isomorphism

$$H^2(\mathcal{J}^\bullet) \cong \operatorname{coker}\left(\bigoplus_{X \in L_2^{\text{trip}}} D(\mathcal{A}_X, \mathbf{m}_X) \xrightarrow{\sum \overline{\psi_X}} \operatorname{coker}(\iota) \right)$$

follows from the tail end of the snake lemma. The statement (1) now follows from the isomorphism $H^1(\mathcal{D}^\bullet) \cong H^2(\mathcal{J}^\bullet)$ and Theorem 1.1.

The ideals $J(H) \cong \langle \alpha_H^{\mathbf{m}(H)} \rangle$ are principal, so are isomorphic to the polynomial ring S (up to a graded shift). The rank of $\bigoplus J(H)$ is $|\mathcal{A}|$ and by the definition of the map ι, we see that the kernel is spanned by the basis elements $[H]$ so that H does not pass through any $X \in L_2^{\text{trip}}$. However, any such hyperplane is a *generic* hyperplane; by Corollary 4.13 the existence of such a hyperplane forces $\mathcal{A}$ to be non-formal. Hence if $\mathcal{A}$ is TF_2, ι is injective. Since $\bigoplus_{X \in L_2^{\text{trip}}} \left[\bigoplus_{H_i < X} J(H_i) \right]$ is a free module of rank $\sum_{X \in L_2^{\text{trip}}} |\mathcal{A}_X|$, we have proved that $\operatorname{coker}(\iota) \cong S^\kappa$. The map ι is surjective if and only if $\kappa = 0$, in which case $H^2(\mathcal{J}^\bullet) = 0$ regardless of the multiplicity $\mathbf{m}$. In this case $\mathcal{A}$ is totally free; by [8] $\mathcal{A}$ is a product of one and two dimensional arrangements, violating the assumption that $\mathcal{A}$ is irreducible. This proves (2).

For (3), notice that, in order for $D(\mathcal{A}, \mathbf{m})$ to be free, the image of $\sum \psi_X$ and the image of ι must span the entire free module $\bigoplus_{X \in L_2^{\text{trip}}} \left[\bigoplus_{H < X} J(H) \right]$. Given (1), the image of ι does not span this entire free module. This means that there are some basis elements $[X, H]$ of degree $\mathbf{m}(H)$ (for some hyperplane H) that remain in $\operatorname{coker}(\iota)$. In order to kill such basis elements, there must be a basis element $\theta_X \in D(\mathcal{A}_X, \mathbf{m}_X)$ of degree $\mathbf{m}(H)$ which does not vanish on α_H. Notice that for a fixed $X \in L_2^{\text{trip}}$, there cannot be two distinct $H, H' \in \mathcal{A}_X$ so that $\deg(\theta_X) = \mathbf{m}(H)$, $\deg(\psi_X) = \mathbf{m}(H')$, with $\theta_X(\alpha_H) \neq 0$ and $\psi_X(\alpha_{H'}) \neq 0$ (see Lemma B.2). Hence there are at most $\#L_2^{\text{trip}}$ derivations (one per $X \in L_2^{\text{trip}}$) that can have the right form to cancel remaining basis elements of $\operatorname{coker}(\iota)$; it follows that if $|\mathcal{A}| + \#L_2^{\text{trip}} < \sum_{X \in L_2^{\text{trip}}}(|\mathcal{A}_X|)$ then $\mathcal{A}$ is totally non-free, proving the first inequality of (3). The equivalent formulation for the inequality follows from the equation $|\mathcal{A}| = r - \#L_2^{\text{trip}} + \sum_{X \in L_2^{\text{trip}}}(|\mathcal{A}_X| - 1)$ from Proposition 6.1. For the final statement of (3), it follows from Lemma A.1 that $\mathcal{A}' \in \mathcal{M}(\mathcal{A})$ is TF_2 on a Zariski open subset of $\mathcal{M}(L(\mathcal{A}))$. Hence on this open set, total non-freeness of $\mathcal{A}'$ follows from the same computation. Moreover, if $\mathcal{A}'$ is in the complement of this open set, $\mathcal{A}'$ is totally non-free by Corollary 4.10. □

Corollary 6.4. *Suppose $\mathcal{A}$ is a TF_2 arrangement with $r(\mathcal{A}) > \#L_2^{trip}$, and suppose $\mathcal{B}$ is an arrangement of rank four. If $L(\mathcal{B})$ has two flats $X, Y \in L(\mathcal{B})$ so that $L(\mathcal{A}) \cong [X, Y]$, then $\mathcal{B}$ is not free.*

Proof. If $L(\mathcal{A})$ is isomorphic to an interval in $L(\mathcal{B})$, then $\mathcal{B}$ has either a closed sub-arrangement or a restriction which is in $\mathcal{M}(L(A))$. In either case, the sub-arrangement or restriction is totally non-free by Proposition 6.2. If $\mathcal{B}$ is free, any closed sub-arrangement is also free. Moreover, the restriction of a free arrangement admits a free multiplicity by Theorem 2.6. Hence $\mathcal{B}$ cannot be free. $\square$

Example 6.5 (Ziegler's Pair)**.** Consider a central arrangement $\mathcal{A}$ of rank three with nine hyperplanes $\alpha_1, \ldots, \alpha_9$ whose lattice has 18 double points and six triple points, explicitly we assume $L_2^{\mathrm{trip}} = \{145, 138, 256, 289, 367, 479\}$. This arrangement can be realized as a line arrangement in $\mathbb{PK}^2$ as the lines extending the edges of a hexagon, along with three lines joining opposite vertices (thus the set L_2^{trip} forms the vertices of the hexagon). Since there is a non-empty Zariski open space of $\mathcal{M}(L)$ on which $\mathcal{A}$ is TF_2 an $\#L_2^{\mathrm{trip}} = 6 > 3 = r(\mathcal{A})$, Proposition 6.2 implies that any $\mathcal{A} \in \mathcal{M}(L)$ is totally non-free. By Corollary 6.4, no $\mathcal{A} \in \mathcal{M}(L)$ can be the restriction of a free arrangement.

This arrangement appears in [40] and [38] as an example of the non-combinatorial behavior of the minimal free resolution of $D(\mathcal{A})$ and the formality of $\mathcal{A}$, respectively. More precisely, it is known (due to Yuzvinsky [38], see also [26, Example 13]) that $\mathcal{A}$ is formal if and only if the points of L_2^{trip} do not lie on a conic in $\mathbb{P}^2$. We may compute that $\mathcal{S}^\bullet$ has the form $0 \to S^3 \xrightarrow{\delta_S^0} S^9 \xrightarrow{\delta_S^1} S^6 \to 0$ if the six points do not lie on a conic and $0 \to S^3 \xrightarrow{\delta_S^0} S^9 \xrightarrow{\delta_S^1} S^5 \xrightarrow{\delta_S^2} S \to 0$ if the six points of L_2^{trip} do lie on a conic (δ_S^1 drops rank).

6.3. **A codimension two incidence graph.** The data in the presentation of $H^2(\mathcal{J}^\bullet)$ in Proposition 6.2 can be combinatorially encoded using the *codimension two incidence graph* of $\mathcal{A}$, which we denote by $G(\mathcal{A})$. The graph $G(\mathcal{A}) = (V, E)$ is a bipartite graph whose vertex set is partitioned as $V = L_2^{\mathrm{trip}} \cup \mathcal{A}$. There is an edge $[X, H]$ between $X \in L_2^{\mathrm{trip}}$ and $H \in \mathcal{A}$ if and only if $H < X$ in $L(\mathcal{A})$ (notice that we do not include codimension two flats which are intersections of just two hyperplanes). Moreover, we define the *reduced* codimension two incidence graph $\overline{G}(\mathcal{A})$ by removing the vertices $H \in V(G(\mathcal{A}))$ of valence one (i.e. removing vertices corresponding to hyperplanes which only pass through a single flat $X \in L_2^{\mathrm{trip}}$).

Now we describe how $G(\mathcal{A})$ and $\overline{G}(\mathcal{A})$ are useful in the context of Proposition 6.2. Referring to the diagram in Figure 6, consider the sub-module N of $\bigoplus\limits_{X \in L_2^{\mathrm{trip}}} \bigoplus\limits_{H<X} J(H)$ generated by the image of ι and the image of $\sum \psi_X$. Since $D(\mathcal{A}_X, \mathbf{m}_X)$ is a free rank two module for every $X \in L_2^{\mathrm{trip}}$, it is generated by two derivations; call these θ_X and ψ_X. Then N is generated by the columns of a matrix we denote $M = M(\theta_X, \psi_X \mid X \in L_2^{\mathrm{trip}})$. The rows of M are naturally indexed by the formal symbols $[X, H]$ corresponding to basis elements of $\bigoplus_{X \in L_2^{\mathrm{trip}}} \bigoplus_{H<X} J(H)$ - equivalently we may assume the rows are indexed by edges of $G(\mathcal{A})$. The columns of M are indexed either by hyperplanes $H' \in \mathcal{A}$ (these represent the image of ι, one for each generator of $\bigoplus_{H \in \mathcal{A}} J(H)$) or pairs $(X', \theta_{X'})$ or $(X', \psi_{X'})$ where $X' \in L_2^{\mathrm{trip}}$ and $\theta_{X'}, \psi_{X'}$ are generators of $D(\mathcal{A}_{X'}, \mathbf{m}_{X'})$ (each pair represents the inclusion of a generator of $D(\mathcal{A}_{X'}, \mathbf{m}_{X'})$). The entries of M are

$$\begin{aligned} M_{[X,H],[H']} &= \begin{cases} 1 & H'=H \\ 0 & H' \neq H \end{cases}, \\ M_{[X,H],[X',\theta_{X'}]} &= \begin{cases} \overline{\theta}_{X'}(\alpha_H) & X'=X \\ 0 & X' \neq X \end{cases}, \\ \text{and } M_{[X,H],[X',\psi_{X'}]} &= \begin{cases} \overline{\psi}_{X'}(\alpha_H) & X'=X \\ 0 & X' \neq X \end{cases}, \end{aligned}$$

where $\overline{\theta}_{X'}(\alpha_H) = \dfrac{\theta_{X'}(\alpha_H)}{\alpha_H^{\mathbf{m}(H)}}$.

Moreover we can associate the non-zero entries of M to *oriented* and labeled edges of $G(\mathcal{A})$; the entry $M_{[X,H],[H]}$ corresponds to the orientation $X \to H$ of $[X,H]$ and the entry $M_{[X,H],[X,\theta_X]}$ corresponds to the orientation $H \to X$ of $[X,H]$, along with the label θ_X on the edge $[X,H]$. If a vertex $H \in G(\mathcal{A})$ has valence one, then the corresponding column of M is a generator of $\bigoplus_{X\in L_2^{\mathrm{trip}}} \bigoplus_{H<X} J(H)$; since we are interested in the cokernel of M we may reduce the matrix M to the matrix $\overline{M}$ whose rows are indexed by pairs $[X,H]$ so that H has valence at least two in $G(\mathcal{A})$. Clearly the rows of $\overline{M}$ are in bijection with edges of the reduced incidence graph $\overline{G}(\mathcal{A})$. Likewise the non-zero entries of $\overline{M}$ correspond to oriented and labeled edges of $\overline{G}(\mathcal{A})$.

By Proposition 6.2, $D(\mathcal{A},\mathbf{m})$ is free if and only if the columns of $M(\theta_X,\psi_X \mid X \in L_2^{\mathrm{trip}})$ generate the free module $\bigoplus_{X\in L_2^{\mathrm{trip}}} \bigoplus_{H<X} J(H)$. As in the proof of Proposition 6.2, only one generator for each $D(\mathcal{A}_X,\mathbf{m}_X)$, $X \in L_2^{\mathrm{trip}}$, can map to a generator of $\bigoplus_{X\in L_2^{\mathrm{trip}}} \bigoplus_{H<X} J(H)$. So we will consider sub-matrices of $\overline{M}$ obtained by choosing only a single generator for each $D(\mathcal{A}_X,\mathbf{m}_X)$. We write $M' = M'(\theta_X \mid X \in L_2^{\mathrm{trip}})$ for the sub-matrix of M formed by choosing a single generator θ_X of each $D(\mathcal{A}_X,\mathbf{m}_X)$, $X \in L_2^{\mathrm{trip}}$. Notice that the columns of M' are now in bijection with the vertices of $\overline{G}$. In the two cases we consider, maximal minors of M' will be obtained by deleting at most one column. Thus the terms of a maximal minor of M' are in bijection with orientations of $\overline{G}$ so that every vertex corresponding to a non-deleted column has exactly one incoming edge. We will use this observation in the next section.

6.4. **Characterization of free multiplicities on TF_2 arrangements.** Using Proposition 6.2 we now characterize free multiplicities on TF_2 arrangements. By Proposition 6.2 and Proposition 6.1 we are restricted to the two cases

- $|\mathcal{A}| = 1 + \sum_{X\in L_2^{\mathrm{trip}}}(|\mathcal{A}_X|-1)$ (equivalently $\mathcal{A}$ is a supersolvable TF_2 arrangement)
- $|\mathcal{A}| = \sum_{X\in L_2^{\mathrm{trip}}}(|\mathcal{A}_X|-1)$

Theorem 6.6 (Free multiplicities on free TF_2 arrangements)**.** *Suppose $\mathcal{A}$ is a free, hence supersolvable TF_2 arrangement. By Proposition 6.1, $\overline{G} = \overline{G}(\mathcal{A})$ is a tree. Then $\mathbf{m}$ is a free multiplicity on $\mathcal{A}$ if and only if there is an orientation of $\overline{G}$ satisfying*

(1) *Every vertex of $\overline{G}$ has at most one incoming edge.*
(2) *The root vertex (no incoming edges) is some $X \in L_2^{trip}$.*
(3) *Given a directed edge $H \to X$, $\mathbf{m}(H)$ is an exponent of $D(\mathcal{A}_X,\mathbf{m}_X)$*

Equivalently, $\mathbf{m}$ is a free multiplicity if and only if there is an ordering $X_1,\dots,X_{r-1}$ of L_2^{trip} and a supersolvable filtration $\mathcal{A}_1 \subset \cdots \subset \mathcal{A}_r$ satisfying

(1) $\mathcal{A}_2 = \mathcal{A}_{X_1}$ *and* $\mathcal{A}_i = \mathcal{A}_{i-1} \cup \mathcal{A}_{X_{i-1}}$
(2) $\mathcal{A}_{X_i} \cap \mathcal{A}_i = \{H_i\}$ *for some* $H_i \in \mathcal{A}$ *(*$H_1, \ldots, H_{r-1}$ *not necessarily distinct)*
(3) $\mathbf{m}(H_i)$ *is an exponent of* $D(X_i, \mathbf{m}_{X_i})$

Proof. By Proposition 6.2 and the preceding discussion, $D(\mathcal{A}, \mathbf{m})$ is free if and only if there are derivations $\theta_X \in D(\mathcal{A}_X, \mathbf{m}_X)$ so that the columns of $\overline{M'} = \overline{M'}(\theta_X \mid X \in L_2^{\text{trip}})$ generate $\bigoplus_{X \in L_2^{\text{trip}}} \bigoplus_{H<X} J(H)$; in other words there should be a maximal minor with determinant equal to a non-zero constant. By Proposition 6.1, we have $|\mathcal{A}| = 1 + \sum_{X \in L_2^{\text{trip}}}(|\mathcal{A}_X| - 1)$ or $|\mathcal{A}| + \#L_2^{\text{trip}} = 1 + \sum_{X \in L_2^{\text{trip}}} |\mathcal{A}_X|$. It follows that the matrix $\overline{M'}$ has one more column than row; so the maximal minors are obtained by deleting a column of $\overline{M'}$. We may assume that the deleted column corresponds to some $X \in L_2^{\text{trip}}$. Since $\overline{G}$ is a tree, an orientation of $\overline{G}$ satisfying that each vertex has at most one incoming edge is equivalent to a choice of root for the tree. This in turn is equivalent to choosing a maximal minor of $\overline{M}$ (leave out the column corresponding to the root). The maximal minor chosen in this way has determinant

$$\prod_{H \to X} \overline{\theta}_X(\alpha_H),$$

where the product is taken over directed edges $H \to X$ in the directed tree $\overline{G}$. This expression is a non-zero constant if and only if $\overline{\theta}_X(\alpha_H)$ is a non-zero constant (equivalently $\theta_X(\alpha_H) = \alpha_H^{\mathbf{m}(H)}$ up to constant multiple) for every directed edge $H \to X$. Since $\mathcal{A}_X$ is not boolean for any $X \in L_2^{\text{trip}}$, we see by Lemma B.2 that $(\mathcal{A}_X, \mathbf{m}_X)$ cannot have an exponent smaller than $\mathbf{m}(H)$, so this is in turn equivalent to $(\mathcal{A}_X, \mathbf{m}_X)$ having an exponent of $\mathbf{m}(H)$ for every directed edge $H \to X$. This proves the first characterization.

We now show the second characterization in terms of supersolvable filtrations is equivalent to the first. Given an orientation of $\overline{G}$, we can build the required filtration by setting X_1 equal to the root vertex and inductively selecting X_{i+1} to satisfy 1) X_i and X_{i+1} are both adjacent to some $H \in \overline{G}$ and 2) $X_i \to H \to X_{i+1}$ is a directed path with respect to the chosen orientation on $\overline{G}$. Conversely, given such a supersolvable filtration, we may orient $\overline{G}$ by taking X_1 to be the root. □

Example 6.7. Suppose $\mathcal{A}$ is defined by $xyz(x-y)(y-z)$ (this is the graphic arrangement corresponding to a four-cycle with a chord). Then $\overline{G}$ consists of two vertices corresponding to the triple points X_1 and X_2 defined by $xy(x-y)$ and $yz(y-z)$, respectively. Clearly $\mathcal{A}$ is a supersolvable TF_2 arrangement. By Theorem 6.6, $(\mathcal{A}, \mathbf{m})$ is free if and only if either $D(X_1, \mathbf{m}_{X_1})$ or $D(X_2, \mathbf{m}_{X_2})$ has an exponent equal to $\mathbf{m}(y)$.

If $\mathbb{K}$ has characteristic zero, this happens if and only if $\mathbf{m}(y) \geq \mathbf{m}(x)+\mathbf{m}(x-y)-1$ or $\mathbf{m}(y) \geq \mathbf{m}(z)+\mathbf{m}(y-z)-1$ (by [33]), which recovers Abe's classification in [2]. In fact Abe's classification has a natural extension to any graphic TF_2 arrangement (these correspond to chordal graphs with two-dimensional clique complex). For instance, suppose $\mathcal{A}$ is defined by $xyzw(x-y)(y-z)(z-w)$. Then $\overline{G}(\mathcal{A})$ has three vertices and Theorem 6.6 combined with the classification in [33] yields that $(\mathcal{A}, \mathbf{m})$ is free if and only if

- $\mathbf{m}(y) \geq \mathbf{m}(x) + \mathbf{m}(x-y) - 1$ and $\mathbf{m}(z) \geq \mathbf{m}(y) + \mathbf{m}(y-z) - 1$ or
- $\mathbf{m}(y) \geq \mathbf{m}(z) + \mathbf{m}(y-z) - 1$ and $\mathbf{m}(z) \geq \mathbf{m}(w) + \mathbf{m}(z-w) - 1$ or
- $\mathbf{m}(y) \geq \mathbf{m}(x) + \mathbf{m}(x-y) - 1$ and $\mathbf{m}(z) \geq \mathbf{m}(w) + \mathbf{m}(z-w) - 1$.

Each of the three possibilities corresponds to a choice of root for $\overline{G}$.

By similar arguments it is not difficult to show that a constant multiplicity of value greater than one is never a free multiplicity on a graphic TF_2 arrangement of rank at least three over a field of characteristic zero. In fact, if the constant multiplicity is free on a graphic arrangement over a field of characteristic zero then it is a product of braid arrangements [15, Theorem 6.6]. In contrast, suppose $\mathbb{K}$ is a field of characteristic p. Then it is straightforward to check (using Saito's criterion), that

$$x^{p^k}\frac{\partial}{\partial x}+y^{p^k}\frac{\partial}{\partial y} \qquad \text{and} \qquad x^{p^{k+1}}\frac{\partial}{\partial x}+y^{p^{k+1}}\frac{\partial}{\partial y}$$

form a basis for the multi-arrangement defined by $x^{p^k}y^{p^k}(x-y)^{p^k}$ (here k is any positive integer). It follows from Theorem 6.6 that the constant multiplicity of value p^k is always free on a graphic TF_2 arrangement over a field of characteristic p. Ziegler [41] has shown that freeness of simple arrangements may also depend on the characteristic of the field.

Example 6.8 (Example 1.2, continued)**.** Consider the arrangement $\mathcal{A}(\alpha,\beta)$ defined by $xyz(x-\alpha z)(x-\beta z)(y-z)$ where $\alpha,\beta\in\mathbb{K}$. This is a TF_2 arrangement with two rank two flats in L_2^{trip}: the flat X_1 defined by $xz(x-\alpha z)(x-\beta z)$ and the flat X_2 defined by $yz(y-z)$. The reduced graph $\overline{G}(\mathcal{A})$ consists of the three vertices H,X_1,X_2 joined by the two edges $[H,X_1]$ and $[H,X_2]$. By Theorem 6.6 a multi-arrangement $(\mathcal{A}(\alpha,\beta),\mathbf{m})$ is free if and only if either $D(\mathcal{A}_{X_1},\mathbf{m}_{X_1})$ or $D(\mathcal{A}_{X_2},\mathbf{m}_{X_2})$ has an exponent of $\mathbf{m}(z)$. Example 1.2 continues the analysis for this multi-arrangement.

Remark 6.9. The characterization in Theorem 6.6 reduces the problem of determining free multiplicities on free TF_2 arrangements to the problem of determining when rank two multi-arrangements have an exponent which is equal to the multiplicity of one of its points, which is a difficult problem in general [34]. Somewhat surprisingly, free multiplicities on non-free TF_2 arrangements admit a complete description, at least in characteristic zero.

Suppose $\mathcal{A}$ is a non-free TF_2 arrangement which admits a free multiplicity. As mentioned earlier, $|\mathcal{A}|=\sum_{X\in L_2^{\text{trip}}}(|\mathcal{A}_X|-1)$ or $|\mathcal{A}|+\#L_2^{\text{trip}}=\sum|\mathcal{A}_X|$. Since $\overline{G}(\mathcal{A})$ is connected (see the proof of Proposition 6.1) and $\overline{G}(\mathcal{A})$ has as many vertices as edges, there is a unique cycle in $\overline{G}(\mathcal{A})$. Write $C=H_0,X_0,H_1,X_1,\ldots,H_{k-1},X_{k-1},H_0$ for this cycle, and let $\alpha_0,\ldots,\alpha_{k-1}$ be the corresponding linear forms to $H_0,\ldots,H_{k-1}$. We observe that the linear forms $\alpha_0,\ldots,\alpha_{k-1}$ must be linearly independent. To see this, define $\mathcal{A}'=\mathcal{A}_{X_0}\cup\mathcal{A}_{X_1}\cdots\cup\mathcal{A}_{X_{k-2}}$. Then $\mathcal{A}'$ has rank k, contains all hyperplanes defined by $\alpha_0,\ldots,\alpha_{k-1}$, and every defining form of $\mathcal{A}'$ is expressible using $\alpha_0,\ldots,\alpha_{k-1}$.

Theorem 6.10 (Free multiplicities on non-free TF_2 arrangements)**.** *Suppose $\mathcal{A}$ is a non-free TF_2 arrangement (over a field of characteristic zero) which admits a free multiplicity. As above, let $C=H_0,X_0,H_1,X_1,\ldots,H_{k-1},X_{k-1},H_0$ be the unique cycle in $\overline{G}=\overline{G}(\mathcal{A})$. Then $\mathbf{m}$ is a free multiplicity on $\mathcal{A}$ if and only if the following conditions are satisfied*

(1) $\mathbf{m}(H)=1$ *for every $H\in\mathcal{A}$ which is not a vertex of C*
(2) *There is an integer $n>0$ so that $\mathbf{m}(H)=n$ for every $H\in\mathcal{A}$ which is a vertex of C*
(3) *There are $B_1,\ldots,B_k\in\mathbb{K}$ satisfying*

- *$B_1 \cdots B_k \neq 1$ and*
- *for every $H \in \mathcal{A}_{X_i} \setminus \{H_i, H_{i+1}\}$ (indices taken modulo k), α_H can be written (up to scalar multiple) as $\alpha_H = \alpha_i + \beta_i^H \alpha_{i+1}$ (indices taken modulo k) for some $\beta_i^H \in \mathbb{K}$ satisfying $(\beta_i^H)^{n-1} = B_i$*

Proof. By Proposition 6.2, we have $|\mathcal{A}| = \sum_{X \in L_2^{\mathrm{trip}}} (|\mathcal{A}_X| - 1)$ or $|\mathcal{A}| + \# L_2^{\mathrm{trip}} = \sum |\mathcal{A}_X|$. So for any choice of θ_X for every $X \in L_2^{\mathrm{trip}}$ the matrix $\overline{M'} = \overline{M'}(\theta_X \mid X \in L_2^{\mathrm{trip}})$ is a square matrix. We find its determinant. A term of $\det(\overline{M'})$ corresponds to an orientation of $\overline{G}$ in which every vertex has exactly one incoming edge. Since $\overline{G}$ has a unique cycle, such an orientation of $\overline{G}$ is determined by an orientation of the cycle (every other edge must be directed 'away' from the cycle). Since there are only two choices of orientation for the cycle C which satisfy that every vertex has exactly one incoming edge, there are only two terms in $\det(\overline{M})$. In fact, if $C = H_0, X_0, H_1, X_1, \ldots, H_{k-1}, X_{k-1}, H_0$,

$$\det(\overline{M}) = \left(\prod_{i=0}^{k-1} \overline{\theta}_{X_i}(\alpha_i) - \prod_{i=0}^{k-1} \overline{\theta}_{X_i}(\alpha_{i+1}) \right) \prod_{(H \to X) \notin C} \overline{\theta}_X(\alpha_H), \tag{1}$$

where the index $i+1$ is taken modulo k and the directed edge $H \to X$ is the unique direction 'away' from the cycle C. From Proposition 6.2, $(\mathcal{A}, \mathbf{m})$ is free if and only if there is a choice of θ_X for every $X \in L_2^{\mathrm{trip}}$ so that the determinant (1) is a non-zero constant. We assume that we have such a choice of $\theta_X, X \in L_2^{\mathrm{trip}}$, and deduce the form for $(\mathcal{A}, \mathbf{m})$ given in the theorem. Lemma B.2 guarantees that $\overline{\theta}_X(\alpha_H) \neq 0$ for any $X \in L_2^{\mathrm{trip}}$ and $H < X$. Now, fixing an arbitrary X_i in the cycle C, we must have $\theta_{X_i}(\alpha_i) = s_i \alpha_i^{\mathbf{m}(\alpha_i)}$ and $\theta_{X_i}(\alpha_{i+1}) = t_i \alpha_{i+1}^{\mathbf{m}(\alpha_{i+1})}$ for some non-zero constants s_i and t_i. Hence $\mathbf{m}(\alpha_i) = \mathbf{m}(\alpha_{i+1}) = \deg(\theta_{X_i})$. Reading around the cycle C, we see that $\mathbf{m}(\alpha_0) = \mathbf{m}(\alpha_1) = \cdots = \mathbf{m}(\alpha_{k-1}) = n$ for some positive integer n, proving (2).

Next again fix an arbitrary X_i in the cycle C and consider the multi-arrangement $(\mathcal{A}_{X_i}, \mathbf{m}_{X_i})$. Since X_i has rank 2, we may assume $(\mathcal{A}_{X_i}, \mathbf{m}_{X_i})$ is defined by $\mathcal{Q}(\mathcal{A}_{X_i}, \mathbf{m}_{X_i}) = x^n y^n \prod_{j=1}^k (x - a_j y)^{m_j}$ for some integer $k \geq 1$ (since $X \in L_2^{\mathrm{trip}}$) and some non-zero constants $a_1, \ldots, a_k$ (we are writing m_j for $\mathbf{m}(x - a_j y)$). Notice that $m_j \leq n$ for all $j = 1, \ldots, k$ since θ_{X_i} has degree n (this is easily seen by applying Lemma B.2). In particular, $(\mathcal{A}_X, \mathbf{m}_X)$ is *balanced* - i.e. $2n \leq |\mathbf{m}_X| = 2n + \sum_{i=1}^k m_i$.

Next, a result of Abe [3, Theorem 1.6] shows that the exponents of a balanced 2-multi-arrangement differ by at most $|\mathcal{A}| - 2 = k$. Write $d_1^{X_i} \geq d_2^{X_i}$ for the exponents of $(\mathcal{A}_{X_i}, \mathbf{m}_{X_i})$, and remember that we are assuming $d_2^{X_i} = \deg(\theta_{X_i}) = n$. From Abe's result we get that $|d_1^{X_i} - d_2^{X_i}| = d_1^{X_i} - n \leq k$, so $d_1^{X_i} \leq n + k$. But $|\mathbf{m}_{X_i}| = 2n + \sum_{j=1}^k m_j = n + d_2^{X_i}$, so $d_2^{X_i} = n + \sum_{i=1}^k m_i \leq n + k$ (this last inequality follows from the previous sentence). Since $m_j \geq 1$ for every j, we must have $m_j = 1$ for each $j = 1, \ldots, k$. Now, applying Lemma B.1 implies that $a_1^{n-1} = \cdots = a_k^{n-1}$. This yields the second bullet point under (3).

As remarked just prior to the statement of Theorem 6.10, $\alpha_0, \cdots, \alpha_{k-1}$ are linearly independent. Change coordinates so that $\alpha_0 = x_0, \ldots, \alpha_{k-1} = x_{k-1}$. Lemma B.1 again yields that the derivation θ_{X_i} has the form $\theta_{X_i} = x_i^n \frac{\partial}{\partial x_i} +$

$B_i x_{i+1}^n \frac{\partial}{\partial x_{i+1}}$. Plugging this into equation (1) yields

$$\det(\overline{M}) = \left(1 - \prod_{i=0}^{k-1} B_i\right) \prod_{(H\to X)\notin C} \overline{\theta}_X(\alpha_H), \tag{2}$$

yielding the first bullet point under (3) since this must be a *non-zero* constant.

Now we prove (1). If $H \in \mathcal{A}$ is not a vertex of C but there is some $X \in C$ so that $H < X$, then $H \in \mathcal{A}_X$ and $\mathbf{m}(H) = 1$ since $H \notin C$. So suppose $H \in \mathcal{A}$ but $H \not< X$ for any $X \in C$. Then $H < X$ for some $X \in L_2^{\text{trip}}$, and $X \notin C$. Then there is a unique H' so that H' is closer to C than X as vertices of $\overline{G}$. Thus $H' \to X$ is a directed edge in any orientation of $\overline{G}$ satisfying that every vertex has a unique incoming edge. Thus $\theta_X(\alpha_H)$ appears in the expression of Equation (2) and $\theta_X(\alpha_H) = \alpha_H^{\mathbf{m}(H)} = \alpha_H$ (up to constant multiple, since we assume the right hand side of Equation (1) is a non-zero constant). It follows from Lemma B.2 that $(A_X, \mathbf{m}_X)$ is simple, i.e. $\mathbf{m}_X \equiv 1$. Hence $\mathbf{m}(H) = 1$ as well.

Finally, suppose $\mathcal{A}$ is a non-free TF_2 arrangement and $(\mathcal{A}, \mathbf{m})$ has the form indicated in the statement of the theorem. Then clearly $\det(\overline{M})$ is a non-zero constant by equation (2), so $(\mathcal{A}, \mathbf{m})$ is free by Proposition 6.2. □

Example 6.11 (Example 1.3, revisited)**.** Consider the arrangement $\mathcal{A}(\alpha, \beta)$ defined by $xyz(x - \alpha y)(x - \beta y)(y - z)(z - x)$, where $\alpha, \beta \in \mathbb{K}$. This is a non-free TF_2 arrangement with three rank two flats in L_2^{trip}: the flat X_0 defined by $xy(x - \alpha y)(x - \beta y)$, the flat X_1 defined by $yz(y - z)$, and the flat X_2 defined by $xz(x - z)$. The reduced graph $\overline{G}(\mathcal{A})$ consists of the cycle $C = \{H_0, X_0, H_1, X_1, H_2, X_2, H_0\}$, where $H_0 = V(x), H_1 = V(y)$, and $H_2 = V(z)$. By Theorem 6.6 the $(\mathcal{A}(\alpha, \beta), \mathbf{m})$ is free if and only if $\mathcal{Q}(\mathcal{A}, \mathbf{m})$ has the form

$$\mathcal{Q}(\mathcal{A}, \mathbf{m}) = x^n y^n z^n (x - \alpha y)(x - \beta y)(y - z)(z - x),$$

where $\alpha^{n-1} = \beta^{n-1} \neq 1$.

6.5. **Further counterexamples to Orlik's conjecture.** In this section we consider the family of arrangements $\mathcal{A}_{r,t}$ with defining polynomial

$$\mathcal{Q}(\mathcal{A}_{r,t}) = x_0 \left(\prod_{i=1}^{r} (x_i^2 - x_0^2)\right) (x_1 - x_2) \cdots (x_{r-1} - x_r)(x_r - t x_1),$$

where $t \neq 0 \in \mathbb{K}$. Write $H_0 = V(x_0)$. The restriction $\mathcal{A}_{r,t}^{H^0}$ has defining polynomial

$$\mathcal{Q}(\mathcal{A}_{r,t}^{H_0}) = \left(\prod_{i=1}^{r} x_i\right) (x_1 - x_2) \cdots (x_{r-1} - x_r)(x_r - t x_1).$$

Ziegler's multi-restriction has the defining polynomial

$$\mathcal{Q}(\mathcal{A}^{H_0}, \mathbf{m}^{H_0}) = \left(\prod_{i=1}^{r} x_i^2\right) (x_1 - x_2) \cdots (x_{r-1} - x_r)(x_r - t x_1)$$

Proposition 6.12. *If $t \neq 1$ and $\mathbb{K}$ has characteristic zero, the arrangement $\mathcal{A}_{r,t}$ satisfies*

1. $(\mathcal{A}_{r,t}^{H_0}, \mathbf{m}^{H_0})$ *is free for $t \neq 0, 1$,*
2. $\mathcal{A}_{r,t}$ *is free if and only if $t = -1$,*

(3) *The minimal free resolution of* $D(\mathcal{A}_{r,t}^{H_0})$ *is a twisted and truncated Koszul complex,* reg $(D(\mathcal{A}_{r,t}^{H_0})) = 3$*, and pdim*$(D(\mathcal{A}_{r,t}^{H_0})) = r-2$ *(the maximum).*

Proof. Write $X_{r,t}$ for $\mathcal{A}_{r,t}^{H_0}$, α_i for x_i $(i = 1, \dots, r)$, β_i for $x_i - x_{i+1}$ $(i = 1, \dots, r-1)$, and β_r for $x_r - tx_1$. The space of all relations on the linear forms of X_r is an r-dimensional space. Write Y_i for the 'triple flat' of codimension two given by the vanishing of the forms $\alpha_i, \alpha_{i+1}, \beta_i$ for $i = 1, \dots, r-1$, and write Y_r for the flat determined by $\alpha_1, \alpha_r, \beta_r$. Clearly $L_2^{\mathrm{trip}} = \{Y_1, \dots, Y_r\}$ and it is not difficult to see that each Y_i contributes one relation to the relation space and they are all linearly independent, hence $X_{r,t}$ is a TF_2 arrangement. Since $\#L_2^{\mathrm{trip}} = r$, the rank of $X_{r,t}$, it follows from Theorem 6.10 that $\mathbf{m}^{H_0}$ is a free multiplicity on $X_{r,t}$, proving (1).

For (2), we use Theorem 2.6. We already have $(\mathcal{A}_{r,t}^{H_0}, \mathbf{m}^{H_0})$ free by (1), so we consider local freeness of $\mathcal{A}_{r,t}$ along H_0. If $t \neq -1$, then the closed sub-arrangement with defining equation

$$(x_1^2 - x_0^2)(x_r^2 - x_0^2)(x_r - tx_1)x_0$$

is not free, so neither is $\mathcal{A}_{r,t}$. So we need to prove local freeness when $t = -1$. The closed sub-arrangements of $\mathcal{A}_{r,-1}$ along H_0 are isomorphic to $A_1 \times A_1 \times A_1, A_1 \times A_2, A_3$ with a hyperplane removed (the *deleted* A_3 arrangement), or A_3. Since these are all free, $\mathcal{A}_{r,-1}$ is free by Theorem 2.6.

For (3), we use the presentation from Proposition 6.2. We consider only the case $\mathbf{m} \equiv 1$. As in Proposition 6.2, write formal symbols $[H]$ (or $[\alpha_H]$) for the generator of $J(H) = \langle \alpha_H \rangle$ and $[X, H]$ (or $[X, \alpha_H]$) for the generator of $J(H)$ inside the direct sum $\bigoplus_{X \in L_2^{\mathrm{trip}}} \bigoplus_{H<X} J(H)$. In the case of $X_{r,t}$, the map $\iota : \bigoplus J(H) \to \bigoplus_{X,H} J(H)$ has the form $\iota([\alpha_i]) = [Y_i, \alpha_i] + [Y_{i+1}, \alpha_i]$ for $i = 1, \dots, r-1$, $\iota([\alpha_r]) = [Y_r, \alpha_r] + [Y_r, \alpha_1]$, and $\iota([\beta_i]) = [Y_i, \beta_i]$. Hence in coker$(\iota)$, we may disregard the generators corresponding to $[Y_i, \beta_i]$ and we can choose generators $[Y_1, \alpha_1], \cdots, [Y_r, \alpha_r]$ with $[Y_2, \alpha_1] = -[Y_1, \alpha_1]$, etc. With this choice of basis, we determine that the map $\sum \overline{\psi_X} : \oplus D(\mathcal{A}_X, \mathbf{m}_X) \to$ coker(ι) is given on $\theta \in D(\mathcal{A}_{Y_1}, \mathbf{m}_{Y_1})$ by $\theta \to \overline{\theta}(\alpha_1)[Y_1, \alpha_1] + \overline{\theta}(\alpha_2)[Y_1, \alpha_2] = \overline{\theta}(\alpha_1)[Y_1, \alpha_1] - \overline{\theta}(\alpha_2)[Y_2, \alpha_2]$, where $\overline{\theta}(\alpha_i) = \theta(\alpha_i)/\alpha_i$ (and similarly for $\theta \in D(\mathcal{A}_{Y_i}, \mathbf{m}_{Y_i}), i > 1$). Thus we may represent the map $\sum \overline{\psi}_X$ by the matrix

$$\begin{array}{c} \\ {[Y_1,\alpha_1]} \\ {[Y_2,\alpha_2]} \\ {[Y_3,\alpha_3]} \\ \vdots \\ {[Y_r,\alpha_r]} \end{array}
\begin{array}{c} \begin{array}{ccccccc} \theta_1 & \upsilon_1 & \theta_2 & \upsilon_2 & \cdots & \theta_r & \upsilon_r \end{array} \\
\begin{pmatrix}
\overline{\theta}_1(x_1) & \overline{\upsilon}_1(x_1) & 0 & 0 & \cdots & -\overline{\theta}_r(x_1) & -\overline{\upsilon}_r(x_1) \\
-\overline{\theta}_1(x_2) & -\overline{\upsilon}_1(x_2) & \overline{\theta}_2(x_2) & \overline{\upsilon}_2(x_2) & \cdots & 0 & 0 \\
0 & 0 & -\overline{\theta}_2(x_3) & -\overline{\upsilon}_2(x_3) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \overline{\theta}_r(x_r) & \overline{\upsilon}_r(x_r)
\end{pmatrix} \end{array}$$

Now, for $i = 1, \dots, r$, $D(\mathcal{A}_{Y_i})$ is generated by the derivations

$$\theta_i = x_i \frac{\partial}{\partial x_i} + x_{i+1} \frac{\partial}{\partial x_{i+1}}$$
$$\upsilon_i = x_i^2 \frac{\partial}{\partial x_i} + x_{i+1}^2 \frac{\partial}{\partial x_{i+1}}$$

for $i = 1, \dots, r-1$ and $D(Y_r)$ is generated by

$$\theta_r = x_r \frac{\partial}{\partial x_r} + x_1 \frac{\partial}{\partial x_1}$$
$$\upsilon_r = x_r^2 \frac{\partial}{\partial x_r} + t x_1^2 \frac{\partial}{\partial x_1}$$

So the above matrix simplifies to

$$M = \begin{array}{c} \\ [Y_1,\alpha_1] \\ [Y_2,\alpha_2] \\ [Y_3,\alpha_3] \\ \vdots \\ [Y_r,\alpha_r] \end{array} \begin{array}{c} \begin{array}{ccccccc} \theta_1 & \upsilon_1 & \theta_2 & \upsilon_2 & \cdots & \theta_r & \upsilon_r \end{array} \\ \begin{pmatrix} 1 & x_1 & 0 & 0 & \cdots & -1 & -tx_1 \\ -1 & -x_2 & 1 & x_2 & \cdots & 0 & 0 \\ 0 & 0 & -1 & -x_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & x_r \end{pmatrix} \end{array}$$

Notice that in $\operatorname{coker}(M)$, the Euler derivations $\theta_1, \dots, \theta_r$ identify all basis elements $[Y_1, \alpha_1], \cdots, [Y_r, \alpha_1]$ to a single basis element. Hence

$$\operatorname{coker}(M) \cong H^2(\mathcal{J}^\bullet) \cong \frac{S(-1)}{\langle x_1 - x_2, x_2 - x_3, \dots, x_{r-1} - x_r, x_r - tx_1 \rangle},$$

where the $S(-1)$ encodes the fact that the degrees of $[Y_i, \alpha_i]$ are all one. Since $t \neq 0, 1$, $H^2(\mathcal{J}^\bullet) \cong S/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of S.

Now, applying the snake lemma to the diagram in Figure 6 and using the fact that ι is injective (see the proof of Proposition 6.2), we get the four-term exact sequence

$$0 \to D(X_{r,t}) \to \bigoplus_{Y \in L_2^{\text{trip}}} D((X_{r,t})_Y, \mathbf{m}_Y) \xrightarrow{M} S(-1)^\kappa \to H^2(\mathcal{J}(X_{r,t})) \to 0,$$

where $S(-1)^\kappa = \operatorname{coker}(\iota)$. Above we noticed this prunes down to

$$0 \to D(X_{r,t}) \to S(-1) \oplus S(-2)^r \xrightarrow{T} S(-1) \to \frac{S}{\mathfrak{m}} \to 0,$$

where $T = \begin{bmatrix} 0 & x_1 - x_2 & \cdots & x_r - tx_1 \end{bmatrix}$. It follows that

$$D(X_{r,t}) \cong S(-1) \oplus K_2(\mathfrak{m})(-1),$$

where $K_2(\mathfrak{m})(-1)$ is the module of second syzygies of $\mathfrak{m}$, twisted by -1. It is well-known that $K_2(\mathfrak{m})$ has $\binom{r}{2}$ generators of degree 2, so $D(X_{r,t})$ is generated by the Euler derivation along with $\begin{pmatrix} r \\ 2 \end{pmatrix}$ generators of degree 3. Its minimal free resolution is given by truncating the Koszul complex at $K_2(\mathfrak{m})$, so it is linear of length $r-2$, the maximum possible. Since the resolution is linear, *reg* $(D(X_{r,t})) = 3$, where *reg* denotes Castelnuovo-Mumford regularity. This completes the proof of (3). □

Remark 6.13. If $t \neq 1$, then the only non-boolean generic flats of $X_{r,t}$ are the obvious ones of rank two corresponding to the closed circuits of length three. Hence the bound on $\operatorname{pdim}(X_{r,t})$ given by Corollary 3.15 is zero, while $\operatorname{pdim}(X_{r,t}) = r-2$. If $t = 1$ then we can see that $\beta_1, \dots, \beta_r$ forms a closed circuit of length r, in which case $\operatorname{pdim}(D(X_{r,1}, \mathbf{m})) \geq r-3$ by Corollary 3.15. In fact, if we introduce the extra variable x_0 and change coordinates by the rule $x_i \to x_i - x_0$, we see that $X_{r,1}$ is the graphic arrangement corresponding to a wheel with r spokes. From [15, Example 7.1], $\operatorname{pdim}(D(X_{r,1}, \mathbf{m})) = r-3$ for any multiplicity $\mathbf{m}$.

$$\begin{tikzcd}
 & \bigoplus\limits_{i=1}^{n} J(H_i) \arrow[r, "\cong"] \arrow[d, "\iota"] & \bigoplus\limits_{i=1}^{n} J(H_i) \arrow[d, "\delta_J^1"] \\
\bigoplus\limits_{X \in L_2^{\text{trip}}} D(\mathcal{A}_X, \mathbf{m}_X) \arrow[r, "\sum \psi_X"] \arrow[d, "\hat{q}"] & \bigoplus\limits_{X \in L_2^{\text{trip}}} \left[\bigoplus\limits_{H_i < X} J(H_i) \right] \arrow[r, "\sum (\delta_J^1)_X"] \arrow[d, "q"] & \bigoplus\limits_{X \in L_2^{\text{trip}}} J_2(\mathcal{A}_X, \mathbf{m}_X) \arrow[d, "\delta_J^2"] \\
\ker(\Delta) \arrow[r, "i"] & \operatorname{coker}(\iota) \cong S^{\kappa} \arrow[r, "\Delta"] & J_3(\mathcal{A}, \mathbf{m})
\end{tikzcd}$$

FIGURE 7. Diagram for Proposition 7.1

7. The case of line arrangements

It is well-known that $D(\mathcal{A})$ may be identified with the module of syzygies on the Jacobian ideal $\operatorname{Jac}(\mathcal{A})$ of the defining polynomial of $\mathcal{A}$; hence $\mathcal{A}$ is free if and only if $\operatorname{Jac}(\mathcal{A})$ is codimension two and Cohen-Macaulay. In this section we show that, for rank three arrangements, $D(\mathcal{A}, \mathbf{m})$ may be identified with potentially higher syzygies of a less geometric object. We use this to give another formulation of Terao's conjecture for lines in $\mathbb{P}^2$.

First, suppose $\mathcal{A}$ is a TF_2 arrangement and consider the diagram in Figure 6. Since ι is injective (see the proof of Proposition 6.2) and $H^1(\mathcal{J}^\bullet) \cong D(\mathcal{A}, \mathbf{m})$, the full snake lemma applied to this diagram yields the exact sequence

$$0 \to D(\mathcal{A}, \mathbf{m}) \to \bigoplus_{X \in L_2^{\text{trip}}} D(\mathcal{A}_X, \mathbf{m}_X) \to S^{\kappa} \to H^2(\mathcal{J}^\bullet) \to 0,$$

where the inclusion $D(\mathcal{A}, \mathbf{m}) \to \bigoplus D(\mathcal{A}_X, \mathbf{m}_X)$ is the sum of the restriction maps $D(\mathcal{A}, \mathbf{m}) \to D(\mathcal{A}_X, \mathbf{m}_X)$ (recall that the isomorphism $D(\mathcal{A}, \mathbf{m}) \cong H^1(\mathcal{J})$ is given by the map $\psi(\theta) = \sum_{H \in L} \theta(\alpha_H)$). By Theorem 1.1, $D(\mathcal{A}, \mathbf{m})$ is free if and only if

$$0 \to D(\mathcal{A}, \mathbf{m}) \to \bigoplus_{X \in L_2^{\text{trip}}} D(\mathcal{A}_X, \mathbf{m}_X) \xrightarrow{\sum \overline{\psi_X}} S^{\kappa} \to 0$$

is a short exact sequence. Hence if $D(\mathcal{A}, \mathbf{m})$ is free we may identify it with the syzygies on a (necessarily non-minimal) set of generators for the free module S^{κ}.

Now suppose $\mathcal{A}$ is rank three, irreducible and totally formal but not TF_2, so $\mathcal{S}^3(\mathcal{A}) = S_3(\mathcal{A}) \neq 0$. We can set up (see Figure 7) a very similar diagram to the one in Figure 6. All maps in the top two rows of Figure 7 are the same as in Figure 6; in particular $\kappa = \sum_{X \in L_2^{\text{trip}}} |\mathcal{A}_X| - |\mathcal{A}|$ just as in Proposition 6.2. The chain complex $\mathcal{J}^\bullet(\mathcal{A}, \mathbf{m})$ appears as the right-most column. The map labeled q is the quotient map. The existence of the bottom right horizontal map $\Delta : \operatorname{coker}(\iota) \to J_3(\mathcal{A}, \mathbf{m})$ follows from the commutativity of the upper right square; furthermore Δ is surjective since δ_J^1 and $\sum (\delta_J^1)_X$ are both surjective. The lower left map $i : \ker(\Delta) \to S^{\kappa}$ is the inclusion and the map $\hat{q}$ is lifted from q in the obvious way.

Proposition 7.1. *Let $\mathcal{A}$ be an essential, irreducible, formal arrangement of rank 3 which is not TF_2. Then*

$$H^2(\mathcal{J}) \cong coker\left(\bigoplus_{X\in L_2^{trip}} D(\mathcal{A}_X,\mathbf{m}_X) \xrightarrow{\hat{q}} \ker(\Delta)\right).$$

and $D(\mathcal{A},\mathbf{m})$ is free if and only if $\hat{q}$ is surjective. Moreover, $D(\mathcal{A},\mathbf{m})$ is free if and only if

$$0 \to D(\mathcal{A},\mathbf{m}) \to \bigoplus_{X\in L_2^{trip}} D(\mathcal{A}_X,\mathbf{m}_X) \xrightarrow{i\circ\hat{q}} S^{\kappa}$$

is exact in the first two positions and coker$(i\circ\hat{q}) = J_2(\mathcal{A},\mathbf{m})$; i.e. the above sequence is a free resolution for $J_2(\mathcal{A},\mathbf{m})$. Moreover, the left-most inclusion of $D(\mathcal{A},\mathbf{m})$ into $\bigoplus D(\mathcal{A}_X,\mathbf{m}_X)$ is given by the sum of natural restriction maps.

Proof. The identification of $H^2(\mathcal{J})$ with coker$(\hat{q})$ follows from a long exact sequence in homology. More precisely, the rows of the diagram in Figure 7 are all exact. Hence we may view this diagram as a short exact sequence of chain complexes; the chain complexes are the columns of the diagram. As we saw in the proof of Proposition 6.2, the map ι is injective so the middle column is exact. Thus the long exact sequence in homology splits into three isomorphisms. The first isomorphism yields $H^1(\mathcal{J}) \cong \ker(\hat{q})$; which we may read as $D(\mathcal{A},\mathbf{m}) \cong \ker(\hat{q})$ ($H^1(\mathcal{J}) \cong D(\mathcal{A},\mathbf{m})$ since $\mathcal{A}$ is essential). The second isomorphisms yields $H^2(\mathcal{J}) \cong$ coker$(\hat{q})$, which is the first statement of the proposition. The third isomorphism yields $H^3(\mathcal{J}) = 0$. Hence by Theorem 1.1, $D(\mathcal{A},\mathbf{m})$ is free if and only if $H^2(\mathcal{J}) = 0$, if and only if coker$(\hat{q}) = 0$.

If $\hat{q}$ is surjective (if and only if $D(\mathcal{A},\mathbf{m})$ is free), then im$(\hat{q}) = \ker(\Delta)$; by our previous identification of $D(\mathcal{A},\mathbf{m})$ with $\ker(\hat{q})$ we have freeness of $D(\mathcal{A},\mathbf{m})$ if and only if the sequence

$$0 \to D(\mathcal{A},\mathbf{m}) \to \bigoplus_{X\in L_2^{\mathrm{trip}}} D(\mathcal{A}_X,\mathbf{m}) \xrightarrow{i\circ\hat{q}} S^{\kappa} \xrightarrow{\Delta} J_3(\mathcal{A},\mathbf{m}) \to 0$$

is exact. Chasing the diagram in Figure 7, and using that the map $D(\mathcal{A},\mathbf{m}) \to \bigoplus J(H)$ is given by $\psi(\theta) = \sum \theta(\alpha_H)$, yields that the left-most inclusion is given by the sum of natural restriction maps, so we are done. □

Given a matrix for Δ in the natural choice of basis, we can identify the columns of Δ with a (often non-minimal) set of generators for $J_3(\mathcal{A},\mathbf{m})$. Thus $\ker(\Delta)$ can be identified with syzygies on this set of generators, which we denote by syz(Δ). In this language, we have the following corollary.

Corollary 7.2. *$D(\mathcal{A},\mathbf{m})$ is free if and only if $\sum_{X\in L_2^{trip}}(i\circ\hat{q})(D(A_X,\mathbf{m}_X))$ generates* syz(Δ).

Remark 7.3. Proposition 7.1 and Corollary 7.2 generalize Theorem 3.16 and Corollary 6.3 of [13], where the corresponding statements are worked out for A_3 multi-braid arrangements.

Now consider the case $\mathbf{m} \equiv 1$, which is the setting of Terao's question of whether freeness of $\mathcal{A}$ is combinatorial. In this case a special role is again played by the Euler derivations in $D(\mathcal{A}_X)$. In terms of corollary 7.2, Euler derivations represent syzygies of degree one, which in turn express redundant generators of $J_3(\mathcal{A})$ (just like $J_2(\mathcal{A})$, $J_3(\mathcal{A})$ is generated in degree one). Write $\overline{D}(\mathcal{A})$ for $D(\mathcal{A})$ modulo the summand generated by the Euler derivation. Then, for $X \in L_2^{\text{trip}}$, $\overline{D}(\mathcal{A}_X) \cong S(-|\mathcal{A}_X|+1)$, as a graded S-module. Also write e for the rank of the free module spanned by the image of the Euler derivations of $D(\mathcal{A}_X, \mathbf{m}_X)$ inside of S^κ. Once we have pruned away the Euler derivations, the chain complex from proposition 7.1 (written as a graded complex of S-modules) becomes

$$0 \to \overline{D}(\mathcal{A}) \to \bigoplus_{X \in L_2^{\text{trip}}} S(-|\mathcal{A}_X|+1) \to S(-1)^{\kappa-e} \to J_3(\mathcal{A}) \to 0, \tag{3}$$

and the first two maps are now *minimal* (matrices for these maps will have no constants other than 0). Since it is shown in Proposition 6.1 that freeness of TF_2 arrangements is combinatorial, Terao's question for line arrangements reduces to:

Question 7.4 (Terao's question for line arrangements)**.** If $\mathcal{A}$ is a line arrangement in $\mathbb{P}^2$ which is not TF_2, is exactness of the chain complex (3) combinatorial?

Example 7.5 (A_3 braid arrangement)**.** For $\mathcal{A} = A_3$ braid arrangement defined by the forms $x, y, z, x-y, x-z, y-z$, $J_3(A_3) = \langle x, y, z, x-y, x-z, y-z\rangle$. The A_3 arrangement has four triple points. The image of the Euler derivations $D(\mathcal{A}_X)$, $X \in L_2^{\text{trip}}$ inside of $S^\kappa = S^{12-6} = S^6$ has rank 3, corresponding to the three redundant generators of $J_3(\mathcal{A})$. Pruning off the Euler derivations yields the chain complex

$$0 \to \overline{D}(\mathcal{A}) \to S(-2)^4 \to S(-1)^3 \to J_3(\mathcal{A}) \to 0,$$

which is exact since the Koszul syzygies among x, y, z are obtained from the non-Euler derivations on $D(\mathcal{A}_X)$, $X \in L_2^{\text{trip}}$. This is not minimal since $D(\mathcal{A})$ has a generator of degree 2 which expresses a relation among the four non-Euler derivations around triple points. Once this generator of degree 2 is pruned off we obtain the Koszul complex resolving $J_3(\mathcal{A})$,

$$0 \to S(-3) \to S(-2)^3 \to S(-1)^3 \to J_3(\mathcal{A}) \to 0.$$

As expected, $D(\mathcal{A})$ is free with exponents $1, 2, 3$ (the generators of degree $1, 2$ were pruned off to produce the minimal resolution).

8. Concluding remarks

We have implemented construction of the chain complexes $\mathcal{J}^\bullet, \mathcal{S}^\bullet, \mathcal{D}^\bullet$ in Macaulay2. Instructions for loading the functions and detailed examples may be found at http://math.okstate.edu/people/mdipasq/ under the Research tab.

So far, we have not studied the behavior of the chain complex $\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})$ under deletion and restriction. In particular, we have the following question.

Question 8.1. Is there a short exact sequence of complexes $0 \to \mathcal{D}^\bullet(\mathcal{A}', \mathbf{m}') \to \mathcal{D}^\bullet(\mathcal{A}, \mathbf{m}) \to \mathcal{D}^\bullet(\mathcal{A}'', \mathbf{m}^*) \to 0$ corresponding to a triple $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ of arrangements (in the sense of [23, Definition 1.14]), where $\mathbf{m}^*$ is the Euler multiplicity [7]?

The main difficulty here is to construct the maps between these chain complexes. Constructing such maps would provide a tight relationship to the addition-deletion theorem of [7]. We also are not aware of any relationships between the chain complex $\mathcal{D}^\bullet(\mathcal{A}, \mathbf{m})$ and the characteristic polynomial of $(\mathcal{A}, \mathbf{m})$ or a supersolvable filtration of $\mathcal{A}$.

References

[1] T. Abe. Deletion theorem and combinatorics of hyperplane arrangements. *ArXiv e-prints*, September 2017.

[2] Takuro Abe. Free and non-free multiplicity on the deleted A_3 arrangement. *Proc. Japan Acad. Ser. A Math. Sci.*, 83(7):99–103, 2007.

[3] Takuro Abe. Chambers of 2-affine arrangements and freeness of 3-arrangements. *J. Algebraic Combin.*, 38(1):65–78, 2013.

[4] Takuro Abe. Divisionally free arrangements of hyperplanes. *Invent. Math.*, 204(1):317–346, 2016.

[5] Takuro Abe, Koji Nuida, and Yasuhide Numata. Signed-eliminable graphs and free multiplicities on the braid arrangement. *J. Lond. Math. Soc. (2)*, 80(1):121–134, 2009.

[6] Takuro Abe, Hiroaki Terao, and Max Wakefield. The characteristic polynomial of a multiarrangement. *Adv. Math.*, 215(2):825–838, 2007.

[7] Takuro Abe, Hiroaki Terao, and Max Wakefield. The Euler multiplicity and addition-deletion theorems for multiarrangements. *J. Lond. Math. Soc. (2)*, 77(2):335–348, 2008.

[8] Takuro Abe, Hiroaki Terao, and Masahiko Yoshinaga. Totally free arrangements of hyperplanes. *Proc. Amer. Math. Soc.*, 137(4):1405–1410, 2009.

[9] Takuro Abe and Masahiko Yoshinaga. Free arrangements and coefficients of characteristic polynomials. *Math. Z.*, 275(3-4):911–919, 2013.

[10] Louis J. Billera. Homology of smooth splines: generic triangulations and a conjecture of Strang. *Trans. Amer. Math. Soc.*, 310(1):325–340, 1988.

[11] Louis J. Billera and Lauren L. Rose. A dimension series for multivariate splines. *Discrete Comput. Geom.*, 6(2):107–128, 1991.

[12] K. A. Brandt and H. Terao. Free arrangements and relation spaces. *Discrete Comput. Geom.*, 12(1):49–63, 1994.

[13] M. DiPasquale, C. A. Francisco, J. Mermin, and J. Schweig. Free and Non-free Multiplicities on the A_3 Arrangement. *ArXiv e-prints*, September 2016.

[14] M. DiPasquale and M. Wakefield. Free Multiplicities on the moduli of X_3. *J. Pure Appl. Algebra*, to appear.

[15] Michael DiPasquale. Generalized splines and graphic arrangements. *J. Algebraic Combin.*, 45(1):171–189, 2017.

[16] Paul H. Edelman and Victor Reiner. A counterexample to Orlik's conjecture. *Proc. Amer. Math. Soc.*, 118(3):927–929, 1993.

[17] Paul H. Edelman and Victor Reiner. Free hyperplane arrangements between A_{n-1} and B_n. *Math. Z.*, 215(3):347–365, 1994.

[18] Anthony V. Geramita and Henry K. Schenck. Fat points, inverse systems, and piecewise polynomial functions. *J. Algebra*, 204(1):116–128, 1998.

[19] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[20] Joseph P. S. Kung and Hal Schenck. Derivation modules of orthogonal duals of hyperplane arrangements. *J. Algebraic Combin.*, 24(3):253–262, 2006.

[21] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[22] Peter Orlik. *Introduction to arrangements*, volume 72 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1989.

[23] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[24] Kyoji Saito. Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(2):265–291, 1980.

[25] Hal Schenck. A spectral sequence for splines. *Adv. in Appl. Math.*, 19(2):183–199, 1997.

[26] Hal Schenck. Hyperplane arrangements: computations and conjectures. In *Arrangements of hyperplanes—Sapporo 2009*, volume 62 of *Adv. Stud. Pure Math.*, pages 323–358. Math. Soc. Japan, Tokyo, 2012.

[27] Hal Schenck and Mike Stillman. Local cohomology of bivariate splines. *J. Pure Appl. Algebra*, 117/118:535–548, 1997. Algorithms for algebra (Eindhoven, 1996).
[28] Henry K. Schenck and Peter F. Stiller. Cohomology vanishing and a problem in approximation theory. *Manuscripta Math.*, 107(1):43–58, 2002.
[29] L. Solomon and H. Terao. A formula for the characteristic polynomial of an arrangement. *Adv. in Math.*, 64(3):305–325, 1987.
[30] R. P. Stanley. Supersolvable lattices. *Algebra Universalis*, 2:197–217, 1972.
[31] Hiroaki Terao. Generalized exponents of a free arrangement of hyperplanes and Shepherd-Todd-Brieskorn formula. *Invent. Math.*, 63(1):159–179, 1981.
[32] Stefan O. Tohăneanu. Topological criteria for k-formal arrangements. *Beiträge Algebra Geom.*, 48(1):27–34, 2007.
[33] Atsushi Wakamiko. On the exponents of 2-multiarrangements. *Tokyo J. Math.*, 30(1):99–116, 2007.
[34] Max Wakefield and Sergey Yuzvinsky. Derivations of an effective divisor on the complex projective line. *Trans. Amer. Math. Soc.*, 359(9):4389–4403, 2007.
[35] Masahiko Yoshinaga. Characterization of a free arrangement and conjecture of Edelman and Reiner. *Invent. Math.*, 157(2):449–454, 2004.
[36] Masahiko Yoshinaga. On the extendability of free multiarrangements. In *Arrangements, local systems and singularities*, volume 283 of *Progr. Math.*, pages 273–281. Birkhäuser Verlag, Basel, 2010.
[37] Sergey Yuzvinsky. Cohomology of local sheaves on arrangement lattices. *Proc. Amer. Math. Soc.*, 112(4):1207–1217, 1991.
[38] Sergey Yuzvinsky. The first two obstructions to the freeness of arrangements. *Trans. Amer. Math. Soc.*, 335(1):231–244, 1993.
[39] Sergey Yuzvinsky. Free and locally free arrangements with a given intersection lattice. *Proc. Amer. Math. Soc.*, 118(3):745–752, 1993.
[40] Günter M. Ziegler. Multiarrangements of hyperplanes and their freeness. In *Singularities (Iowa City, IA, 1986)*, volume 90 of *Contemp. Math.*, pages 345–359. Amer. Math. Soc., Providence, RI, 1989.
[41] Günter M. Ziegler. Matroid representations and free arrangements. *Trans. Amer. Math. Soc.*, 320(2):525–541, 1990.

Appendix A. The moduli space of an arrangement

In this appendix we briefly summarize the construction of the moduli space of a lattice over an algebraically closed field $\mathbb{K}$. Given the intersection lattice L of some central arrangement $\mathcal{A} \subset V \cong \mathbb{K}^\ell$ with n hyperplanes,we obtain the *moduli space* of L in the following steps:

(1) Fix an ordering $H_1, \dots, H_n$ of the hyperplanes of $\mathcal{A}$. Then each flat $X \in L$ can be identified with the tuple of integers $i_1, \dots, i_j$ where $H_{i_s} < X$ for every $s = 1, \dots, j$.
(2) Let M be an $n \times \ell$ coefficient matrix of variables and $\mathbb{K}[M]$ the polynomial ring in these variables. The rows of M correspond to the hyperplanes $H_1, \dots, H_n$, in order.
(3) Suppose the flat $X \in L_k$ is defined by hyperplanes $H_{i_1}, \dots, H_{i_j}$, with $j > k$. Then the $(k+1) \times (k+1)$ minors of the submatrix of M formed by the rows $i_1, \dots, i_j$ must all vanish. Let $I \subset \mathbb{K}[M]$ be the radical of the ideal generated by all of these minors for all flats $X \in L$.
(4) Now let $\mathcal{B}$ be the set of all possible tuples of ℓ hyperplanes which intersect in only the origin. Each tuple in $\mathcal{B}$ gives rise to an $\ell \times \ell$ sub-matrix of M whose determinant must not vanish. Let J be the principal ideal generated by the product of all of these determinants.

(5) The quasi-affine variety $\mathcal{V} = \mathcal{V}(L) = V(I) \setminus V(J) \subset \mathcal{M}$, endowed with the Zariski topology, corresponds to coefficient matrices of hyperplane arrangements with intersection lattice L.
(6) Since the correspondence between a coefficient matrix and a hyperplane arrangement is not one-to-one, the moduli space $\mathcal{M}(L)$ of L is obtained from $\mathcal{V}(L)$ by quotienting out by the action of scaling rows of M and a changing coordinates in V.

A property of an arrangement $\mathcal{A}$ is *combinatorial* if it can be determined from its lattice; equivalently if the property holds for all $\mathcal{A}' \in \mathcal{M}(L(\mathcal{A}))$. One of the key open questions in the theory of arrangements (posed by Terao), is whether freeness of arrangements is combinatorial. Yuzvinsky [39] has shown that free arrangements with intersection lattice L form a Zariski open subset of $\mathcal{M}(L)$. It is not difficult to show that a similar condition holds for totally formal arrangements.

Lemma A.1. *If $\mathcal{A}$ is an essential and totally formal arrangement then rank$(\mathcal{S}^i(\mathcal{A}))$ is determined by L for every i. Moreover, the set of essential totally formal arrangements with intersection lattice L is a Zariski open set in $\mathcal{M}(L)$.*

Proof. The arrangement $\mathcal{A}$ is essential and totally formal if and only if $\mathcal{S}^\bullet$ is exact (see Corollary 4.8). Since $\mathcal{S}^k(\mathcal{A}) = \bigoplus_{X \in L_k} S^k(\mathcal{A}_X)$, it suffices to show inductively that $\mathrm{rk}(S^k(\mathcal{A}_X))$ is determined from $L(\mathcal{A}_X)$ for $k = \mathrm{rk}(X)$. If $X \in L(\mathcal{A})$ has rank one, then $\mathrm{rk}\, S^1(\mathcal{A}_X) = 1$. Now the result follows inductively on the rank of $\mathcal{A}_X$, using the Euler characteristic of $\mathcal{S}^\bullet(\mathcal{A}_X)$. See also Remark 4.12.

Now decompose $\mathcal{V}(L)$ into its irreducible components $\mathcal{V}(L) = \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_k$; algebraically, this corresponds to a prime decomposition $I = P_1 \cap P_2 \cdots \cap P_k$ (recall I is radical) where $\mathcal{V}_i = V(P_i) \setminus V(J)$. Fix a component $\mathcal{V}_i$ of $\mathcal{V}(L)$ and work in its coordinate ring $R = \mathbb{K}[M]/P_i$. In other words, we consider an arrangement $\mathcal{A}$ whose coefficient matrix has entries in the integral domain R. By Lemma 5.6 the differentials of the chain complex $\mathcal{S}^\bullet(\mathcal{A})$ (equivalently the differentials of $\mathcal{R}_\bullet$) are elements of the rational function field $K = \mathrm{frac}(R)$. By the first statement, we see that the conditions for $\mathcal{A}_X$ to be k-formal for every $2 \le i \le r(X) - 1$ and every $X \in L$ are finitely many maximal rank conditions on the differentials $\delta^i_{S,X}$ for $\mathcal{S}^\bullet(\mathcal{A}_X)$. Since maximal rank conditions are given by the non-vanishing of certain minors, this shows that there are finitely many rational functions in K that should not vanish if $\mathcal{A}$ and all its closed sub-arrangements are to be k-formal for every k. Lifting this back to R by considering numerators and denominators gives the result for $\mathcal{V}(L)$. Since the determinants in question are multi-homogeneous in the row variables and quotienting by coordinate changes amounts to determining a scalar value for certain variables, this descends to the moduli space $\mathcal{M}(L)$. $\square$

Remark A.2. For a rank three arrangement, the condition to be formal is expressed by the non-vanishing of a maximal rank minor of the δ^2_S differential. Example 6.5 shows that the ranks of the free modules in $\mathcal{S}^\bullet(\mathcal{A})$ are not combinatorial, and that the condition to be totally formal can be non-trivial. For Example 6.5, it can be shown that, aside from the polynomials determining the lattice structure, there is a single irreducible quadratic in the coefficients of the forms of $\mathcal{A}$ whose non-vanishing determines formality.

Appendix B. Two lemmas for multi-arrangements of points in $\mathbb{P}^1$

In this appendix we collect two simple lemmas for multi-arrangements of points in $\mathbb{P}^1$. The first may also be found in [34], in slightly less generality.

Lemma B.1. *Let n be a positive integer and $(\mathcal{A}, \mathbf{m})$ a muli-arrangement of $k + 2 \leq n + 2$ points in $\mathbb{P}^1$ with $Q(\mathcal{A}, \mathbf{m}) = x^n y^n \prod_{i=1}^k (x - a_i y)$. Then $(\mathcal{A}, \mathbf{m})$ has exponents $(n, n + k)$ if and only if $a_1, \dots, a_k$ are distinct $(n-1)$st roots of a non-zero complex number β. In this case, the derivation of degree n has the formula $\theta = x^n \frac{\partial}{\partial x} + \beta y^n \frac{\partial}{\partial y}$.*

Proof. It is straightforward to check that $\theta \in D(\mathcal{A}, \mathbf{m})$ under the conditions of the lemma, so $(\mathcal{A}, \mathbf{m})$ has exponents $(n, n+k)$ if $a_1, \dots, a_k$ are distinct roots of β. Now suppose that there is a derivation $\theta \in D(\mathcal{A}, \mathbf{m})$ of degree n. This corresponds to a syzygy of degree n on the columns of the matrix M from Example 5.7. Hence there exist constants A, B and polynomials $G_1, \dots, G_k$ (of degree $n - 1$) so that

$$Ax^n - Ba_i y^n + G_i(x - a_i y) = 0$$

for every $i = 1, \dots, k$. Dividing through by A and setting $\beta = B/A, \gamma_i = -G_i/A$ yields $x^n - \beta a_i y^n = \gamma_i(x - a_i y)$; hence $a_i^n - \beta a_i = 0$, or $a_i^{n-1} = \beta$. Since this holds for every $i = 1, \dots, k$, the lemma is proved. □

Lemma B.2. *Suppose $(\mathcal{A}, \mathbf{m})$ is a multi-arrangement of $k+2$ points in $\mathbb{P}^1$ defined by forms $\alpha_1, \dots, \alpha_{k+2}$. Suppose that, for some $1 \leq j \leq k+2$, $\theta \in D(\mathcal{A}, \mathbf{m})$ satisfies that $\theta(\alpha_j) = \alpha_j^{\mathbf{m}(\alpha_j)}$ (up to multiplication by a constant). If $\mathcal{A}$ is not boolean, then $\theta(\alpha_i) \neq 0$ for all $i = 1, \dots, k + 2$.*

Proof. Without loss of generality, suppose $\theta(\alpha_2) = 0$ and $\theta(\alpha_1) = \alpha_1^{\mathbf{m}(\alpha_1)}$. Changing coordinates, we may assume $\alpha_1 = x$ and $\alpha_2 = y$. Write $\theta = f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y}$ and let $d = \deg(\theta)$. Since $\theta(x) = x^d$ and $\theta(y) = 0$, $f = x^d$ and $g = 0$. Any other α_j has the form $x + a_j y$ for some non-zero constant a_j; thus we have $\theta(\alpha_j) = x^d$. Since $\theta \in D(\mathcal{A}, \mathbf{m})$, we must have $\alpha_j \mid x^d$, a contradiction unless $\mathcal{A}$ is boolean. □