# Geometric Realizations of the Space of Splines on Simplicial Complexes

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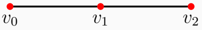
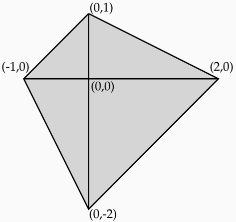
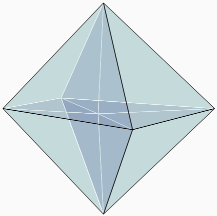
# Spline (or piecewise polynomial) functions

- For a pure $d$-dimensional simplicial complex $\Delta \subset \mathbb{R}^d$



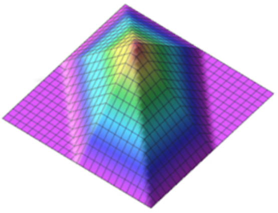
- The space $C^r(\Delta)$ of **splines** is the set of all $C^r$-continuous functions $f : \Delta \mapsto \mathbb{R}$ such that $f|_\sigma$ to each maximal simplex $\sigma$ is a real polynomial.
- The set $C^r(\Delta)$ is a vector space over $\mathbb{R}$.
  One would like to find the dimension and a basis for each of the subspaces $C^r_k(\Delta)$ of elements of degree at most $k$ ...
- Additionally, $C^r(\Delta)$ forms a ring under pointwise multiplication.

➡ What is the **ring structure** of $C^r(\Delta)$? **geometric interpretations**?

# The algebra of continuous splines

- Suppose $v_0, \ldots, v_n$ are the vertices of the simplicial complex $\Delta$.
- Let $Y_i$ be the unique piecewise linear function on $\Delta$ defined by $Y_i(v_j) = \delta_{ij}$ the Kronecker delta.

Then $Y_1, \ldots, Y_m$ form a basis for $C^0_1(\Delta)$ as a real vector space (the Courant functions on $\Delta$).



➡ They generate $C^0(\Delta)$ as an $\mathbb{R}$-algebra, and

$$C^0(\Delta) \cong A_\Delta/(Y_0 + \cdots + Y_n - 1)$$

where $A_\Delta$ is the **face ring** of $\Delta$

$$A_\Delta = \mathbb{R}[Y_0, \ldots, Y_n]/I_\Delta,$$

with $I_\Delta$ is the monomial ideal generated by the products $Y_{i_1} \cdots Y_{i_j}$ such that $\{v_{i_1}, \ldots, v_{i_j}\}$ is not a face of $\Delta$.

# Example: spline space $C^0(\Delta)$ and face ring $A_\Delta$

- For $\Delta \subset \mathbb{R}^2$ with vertices $v_1, \ldots, v_5$,
  The Stanley–Reisner ring

  $$A_\Delta = \mathbb{R}[Y_0, Y_1, Y_2, Y_3, Y_4]/I_\Delta$$

  with $\quad I_\Delta = (Y_1 Y_3, Y_2 Y_4)$.

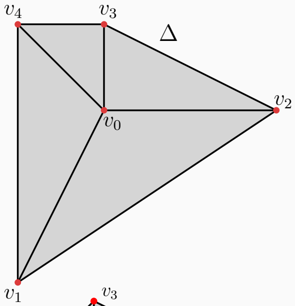
- The spline space:

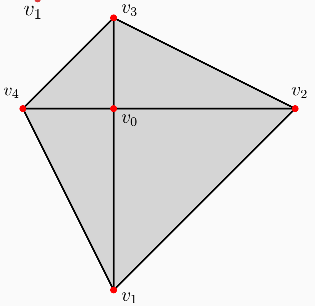
  $$C^0(\Delta) \cong A_\Delta/(Y_0 + \cdots + Y_4 - 1).$$

- If we "homogenize" $\Delta$ then

  $$C^0(\hat{\Delta}) \cong A_\Delta.$$

- It is known [Bruns–Gubeladze] that if two simplicial complexes have isomorphic Stanley–Reisner rings, then they are themselves isomorphic.



# Affine Stanley-Reisner rings

- Identify $Y_i$ with the Courant (hat) function at $v_i$ and extending by linearity $\sum_i Y_i = 1$, and

$$A_\Delta / \left( \sum_i Y_i - 1 \right) \cong C^0(\Delta)$$

is called the affine Stanley-Reisner ring of $\Delta$.

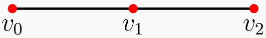
- Then

$$\mathrm{Spec}\big(C^0(\Delta)\big) = \mathrm{Spec}\big(A_\Delta\big) \cap Z\left( \sum_i Y_i - 1 \right) \subset \mathbb{A}^n_{\mathbb{R}} = \mathbb{R}^n$$

and the points that have non-negative coordinates, give a model of $\Delta$.

## Example ($d = 1$):

Let $\Delta$ be a one-dimensional simplicial complex with three vertices $v_0, v_1, v_2 \in \mathbb{R}$, and assume $v_0 < v_1 < v_2$.




We have:

$$C^0(\Delta) = A_\Delta/(\textstyle\sum_{i=1}^{3} Y_i - 1) = \mathbb{R}[Y_0, Y_1, Y_2]/(Y_0 Y_2, Y_0 + Y_1 + Y_2 - 1)$$

$$\mathrm{Spec}(C^0(\Delta)) = Z(Y_0, Y_1 + Y_2 - 1) \cup Z(Y_2, Y_0 + Y_1 - 1) \subset \mathbb{R}^3$$

The segments of these two lines contained in the positive octant mimic the two 1-faces of $\Delta$, and they intersect transversally.

# Subalgebras of the Stanley-Reisner ring

- Consider $\cdots \subseteq C^r(\Delta) \subseteq C^1(\Delta) \subseteq C^0(\Delta) = A_\Delta / \left(\sum Y_i - 1\right)$.
- The diagram is commutative and exact

$$\begin{array}{ccccccc}
0 \longrightarrow & C^r(\Delta) & \longrightarrow & \bigoplus\limits_{\sigma \in \Delta_d} R & \xrightarrow{\partial^r} & \bigoplus\limits_{\substack{\tau = \sigma_i \cap \sigma_j \\ \sigma_i \prec \sigma_j}} R/\ell_\tau^{r+1} \\
& \downarrow i & & \downarrow \oplus B_\sigma & & \downarrow \oplus B_\sigma \\
0 \longrightarrow & \ker(\Psi^r) & \longrightarrow & \bigoplus\limits_{\sigma \in \Delta_d} A_\sigma & \xrightarrow[\sigma_i \prec \sigma_j]{\Psi^r} & \bigoplus\limits_{\substack{\tau = \sigma_i \cap \sigma_j \\ \sigma_i \prec \sigma_j}} A_{\sigma_i}/B_{\sigma_i}(\ell_\tau^{r+1})
\end{array}$$

If $\sigma_i \cap \sigma_j = \tau$ then $\partial^r(f_1, \ldots, f_m)|_\tau = f_i - f_j$ in $R/\ell_\tau^{r+1}$ and,

$$\Psi^r(f_{\sigma_1}, \ldots, f_{\sigma_m})|_\tau = f_{\sigma_i} - B_{\sigma_j \sigma_i}(f_{\sigma_j}) \text{ in } A_\sigma / B_{\sigma_i}(\ell_\tau^{r+1}).$$

- The ring $A_\Delta$ is related to $C^r(\Delta)$ by the inclusion: $\Phi : A_\Delta \to \bigoplus\limits_{\sigma \in \Delta_d} A_\sigma$

defined by $\Phi(Y_i) = \begin{cases} 0 & \text{if} \quad v_i \notin \sigma \\ X_i^\sigma & \text{if} \quad v_i \in \sigma. \end{cases}$

➡ $F \in A_\Delta / (\sum_i Y_i - 1)$ is an element of $C^r(\Delta) \Leftrightarrow \Psi^r(\Phi(F)) = 0$ [Schenck].

# Trivial splines in $A_\triangle$

- For a simplicial complex $\Delta \subset \mathbb{R}^d$, set

$$H_j := v_{1,j} Y_1 + \cdots + v_{n,j} Y_n, \quad \text{for } j = 1, \ldots, d$$

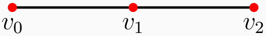
and $H_{d+1} := Y_1 + \cdots + Y_n$, where $v_i = (v_{i,1}, \ldots, v_{i,d})$.

- Then $H_j$ is equal to the $j$-th coordinate function on $\Delta \subset \mathbb{R}^d$, and so

$$\mathbb{R}[H_1, \ldots, H_d] \subseteq C^r(\Delta)$$

is the subring of trivial splines.

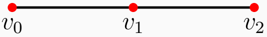
**Example:** $H_1 := v_0 Y_0 + v_1 Y_1 + v_2 Y_2$ is the trivial spline.




In fact, $H_1(x) = x$ for any point $x \in \Delta$, and $\mathbb{R}[H_1] \subseteq C^r(\Delta)$, for any $r \geq 0$.

# Generators of $C^r(\Delta)$ for $d = 1$

- In the case




also $Y_1^{r+1}$ and $Y_3^{r+1}$ correspond to elements in $C^r(\Delta)$.

➡ In fact: $C^r(\Delta) \cong \mathbb{R}[H, Y_1^{r+1}]/(Y_1 Y_3, Y_1 + Y_2 + Y_3 - 1)$.

On the other hand, $H_1 - v_1 = (v_0 - v_1)Y_0 + (v_1 - v_2)Y_2$.

Consider the map

$$\varphi_r : \mathbb{R}[y_0, y_1, y_2] \to \mathbb{R}[Y_0, Y_1, Y_2]/(Y_0 Y_2, Y_0 + Y_1 + Y_2 - 1)$$

$$\varphi(y_1) = H_1, \quad \varphi(y_0) = \big((v_0 - v_1)Y_0\big)^{r+1},$$
$$\varphi(y_2) = \big((v_1 - v_2)Y_2\big)^{r+1}$$

Then, $\mathrm{Im}(\varphi_r) \cong C^r(\Delta)$ and $\ker(\phi_r) = (y_0 y_2,\ y_0 + y_2 - (y_1 + u_2)^{r+1})$.

➡ $\mathrm{Spec}(C^r(\Delta)) = Z(y_0, y_2 - (y_1 - v_1)^{r+1}) \cup Z(y_2, y_0 - (y_1 - v_1)^{r+1})$.

# Geometric realization of $C^r(\Delta)$

- Hence $C^r(\Delta) \cong \mathbb{R}[y_0, y_1, y_2]/(y_0 y_2, y_0 + y_2 - (y_1 - v_1)r + 1)$, and

$$\mathrm{Spec}\big(C^r(\Delta)\big) = Z(y_0, y_2 - (y_1 - v_1)^{r+1}) \cup Z(y_2, y_0 - (y_1 - v_1)^{r+1})$$

- For $r \geq 1$, both curves have the $y_1 - v_1$ line as tangent at their point of intersection (the origin), and the tangent intersects each curve with multiplicity $r + 1$.

# The local spline ring geometric description

Let $\Delta$ be a (general) $d$-dimensional simplicial complex consisting of two d-simplices intersecting in a $(d-1)$-simplex.

Then we can realize $\mathrm{Spec}(C^r(\Delta)) \subset \mathbb{R}^{d+2}$ as the union of two smooth $d$-dimensional varieties $V_1$ and $V_2$ intersecting along a linear $(d-1)$-dimensional space $L$, such that $V_1$ and $V_2$ have the same $d$-dimensional linear space $T$ as tangent space at each point of $L$ and such that $V_i$ and $T$ have order of contact $r+1$ at each point of $L$.

# Idea of the proof

- We have $H_j = v_{0,j} Y_0 + \cdots + v_{d+1,j} Y_{d+1}$ for $j = 1, \ldots, d$, where $v_t = (v_{t,1}, \ldots, v_{t,d}) \in \mathbb{R}^d$ are the vertices of $\Delta$.
- Let $c_0 = \det M_\sigma$ and $c_{d+1} = \det M_{\sigma'}$.
- Define $F := c_0 Y_0 + c_{d+1} Y_{d+1}$, which is a trivial spline on $\Delta$ and therefore, $F = u_1 H_1 + \cdots + u_{d+1} H_{d+1} \quad$ for $u_1, \cdots, u_{d+1} \in \mathbb{R}$.
- Notice that $(c_0 Y_0)^{r+1} + (c_{d+1} Y_{d+1}^{r+1}) = F^{r+1}$.
- Define the map $\varphi_r \colon \mathbb{R}[y_0, \ldots, y_{d+1}] \to \mathbb{R}[Y_0, \ldots, Y_{d+1}] / \big(Y_0 Y_{d+1}, \sum_i Y_i - 1\big)$ by

$$\varphi_r(y_j) = \begin{cases} (c_j Y_j)^{r+1} & \text{for } j = 0, d+1, \\ H_j & \text{for } j = 1, \ldots, d. \end{cases}$$

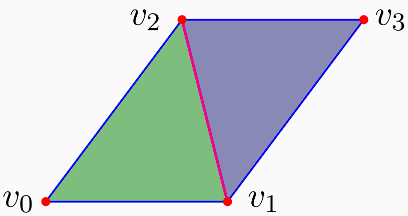
- Then $C^r(\Delta) \cong \mathbb{R}[H_1, \ldots, H_{d+1}, Y_0^{r+1}] / \big(Y_0 Y_{d+1}, \sum_{i=0}^{d+1} Y_i - 1\big)$ implies $\operatorname{Im}(\varphi_r) \cong C^r(\Delta)$.
- $\ker(\varphi_r) = \big(y_0 y_{d+1},\ y_0 + y_{d+1} - \big(\sum_i u_i y_i + u_{d+1}\big)^{r+1}\big)$.

# Geometric realization

Thus, for $\Delta = \sigma \cup \sigma \subset \mathbb{R}^d$:

$$\mathrm{Spec}(C^r(\Delta)) = Z\big(y_0, y_{d+1} - (u_1 y_1 + \cdots + u_d y_d + u_{d+1})^{r+1}\big) \cup Z\big(y_{d+1}, y_0 - (u_1 y_1 + \cdots + u_d y_d + u_{d+1})^{r+1}\big).$$

**Example:** In the case $d = 2$, each $V_i$ is a 2-dimensional variety in a 4-dimensional space. We have $F = \det M_\sigma Y_0 + \det M_{\sigma'} Y_3$. The matrix of the edge $\tau = \sigma \cap \sigma'$

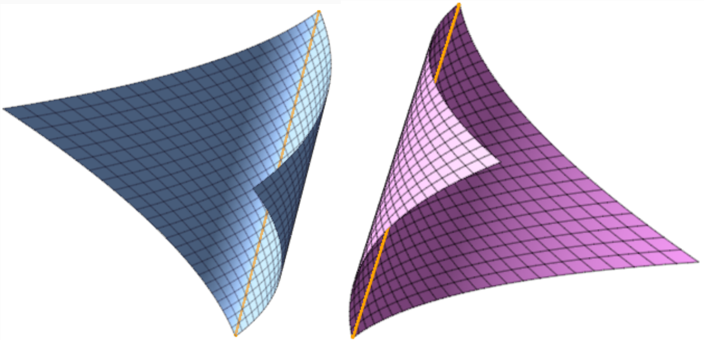



$$M_\tau = \begin{pmatrix} v_{1,1} & v_{1,2} & 1 \\ v_{2,1} & v_{2,2} & 1 \end{pmatrix}$$

leads to $u_1 = v_{1,2} - v_{2,1}$,
$u_2 = -(v_{1,1} - v_{2,1})$,
$u_3 = v_{1,1} v_{2,2} - v_{1,2} v_{2,1}$,
and $u_1 H_1 + u_2 H_2 + u_3 H_3 = F$.

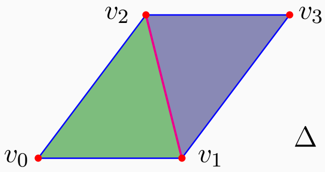
# Example ($d = 2$)

- $\mathrm{Spec}(C^r(\Delta)) =$
  $Z(y_0, y_3 - (u_1 y_1 + u_2 y_2 + u_3)^{r+1}) \cup Z(y_3, y_0 - (u_1 y_1 + u_2 y_2 + u_3)^{r+1})$.

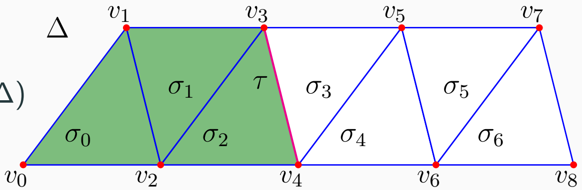


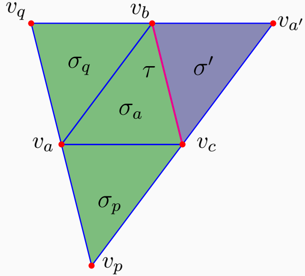
- The intersection of these surfaces is the line $Z(y_0, y_3, u_1 y_1 + u_2 y_2 + u_3)$.
- The plane $Z(y_0, y_3)$ is the tangent plane to both surfaces at all points of their line of intersection. The intersection of this tangent plane and each surface is the line, with multiplicity $r + 1$.

# Generators of $C^r(\Delta)$ as a ring: shellable triangulations




$Y_0^{r+1}, \quad Y_3^{r+1} \in C^r(\Delta)$



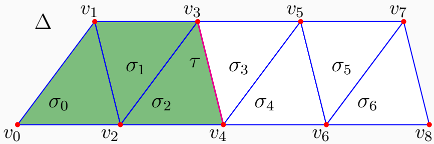

$Y_0^{r+1}, \quad Y_8^{r+1} \in C^r(\Delta)$,
but $Y_2^{r+1} \notin \mathbb{C}^r(\Delta)$.

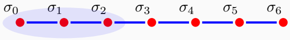
$Y_q^{r+1} \in C^r(\Delta)$,
but $Y_a^{r+1} \notin \mathbb{C}^r(\Delta)$.

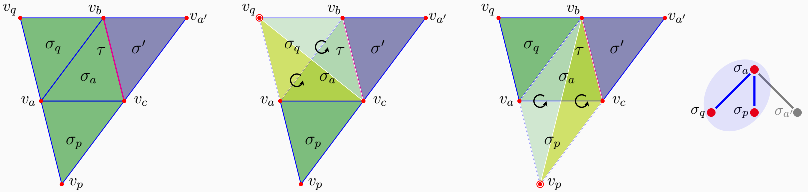
# Generators of the ring $C^r(\Delta)$: shellable triangulations

For any given 1-*shellable* triangulation $\Delta$ with $m$ interior edges, there is a set of linearly independent polynomials $L_1, \ldots, L_m \in A_\triangle$ of degree 1, such that each $L_i^{r+1}$ corresponds in $C^0(\Delta)$ to a $C^r$-continuous non-trivial spline on $\Delta$, and

$$C^r(\Delta) \cong \mathbb{R}[H_1, H_2, H_3, L_1^{r+1}, \ldots, L_m^{r+1}] / \left( I_\Delta, \sum_{i=1}^n Y_i - 1 \right).$$

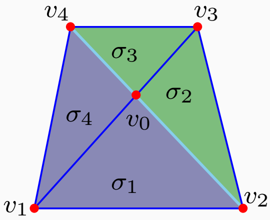


# Triangulations with interior vertices

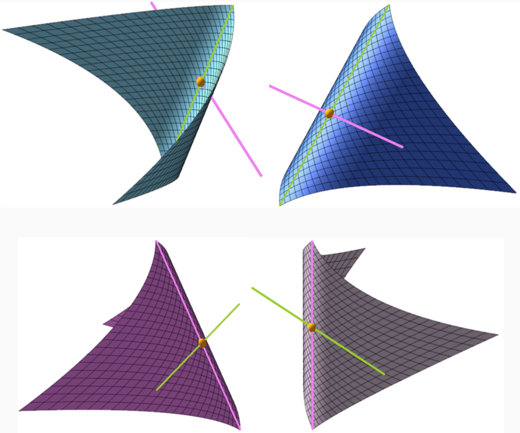
For quadrilateral triangulated by its two diagonals $C^r(\Delta) \cong \mathbb{R}[H_1, H_2, H_3, Y_1^{r+1}, Y_2^{r+1}]/(Y_1 Y_3, Y_2 Y_4, 1 - (Y_0 + Y_1 + Y_2 + Y_3 + Y_4))$.




Similarly as before, there is a map $\varphi_r : \mathbb{R}[y_0, \ldots, y_5] \to \mathbb{R}[Y_0, \ldots, Y_4]/\left(Y_1 Y_3, Y_2 Y_4, \sum_{i=0}^{4} Y_i - 1\right)$ such that $\mathrm{Im}(\varphi_r) \cong C^r(\Delta)$ and $y_1 + y_3 - (u_1 y_0 + u_2 y_5 + u_3)^{r+1}$ and $y_2 + y_4 - (u_1' y_0 + u_2' y_5 + u_3')^{r+1}$ are in $\ker(\varphi_r)$.

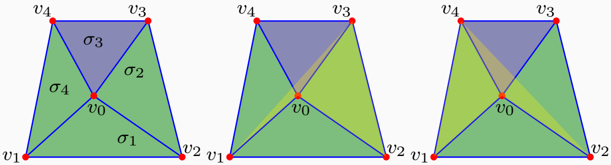
# Interior vertices

- $\mathrm{Spec}(C^r(\Delta)) =$
  $Z(y_1, y_3 - (u_1 y_0 + u_2 y_5 + u_3)^{r+1}) \cup Z(y_3, y_1 - (u_1 y_0 + u_2 y_5 + u_3)^{r+1}) \cup$
  $Z(y_2, y_4 - (u'_1 y_0 + u'_2 y_5 + u'_3)^{r+1}) \cup Z(y_4, y_2 - (u'_1 y_0 + u'_2 y_5 + u'_3)^{r+1})$

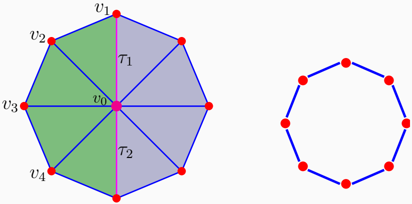


# Dual graph with cycles

For $\Delta$ a generic triangulation with 5 vertices, $C^1(\Delta)$ is generated as a subring of $C^0(\Delta)$ by $H_0, H_1, H_2, S, T$, where $S$ is a (nontrivial) quadratic spline, and $T$ is linear (syzygy).




A similar proposition holds for triangulations whose dual graph is a cycle.



# Final remarks and references

A natural next step is to consider a 2-dimensional $\Delta$ and

(i) find a geometric realization for the space of splines on simplices meeting whose dual graph is a tree,

(ii) and study the generators and geometric realizations in the case of generic simplices with interior vertices. Particularly, the role of the syzygies as elements of the Stanley-Reisner ring.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**Thank you!**