Splines on Lattices and Equivariant Cohomology of Certain Affine Springer Fibers

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Outline

I. Definition of splines

II. Splines on lattices

III. Equivariant cohomology of certain affine Springer fibers

Splines

- Fix a ring R and a graph $G = (V, E)$
- Fix a function $\alpha : E \to \{$ ideals in $R\}$
- The ring of splines over G and α is

$$R_{G,\alpha} = \left\{ p \in R^{|V|} : \begin{array}{c} \text{for each edge } uv \\ \text{the difference } p_u - p_v \in \alpha(uv) \end{array} \right\}$$

Basic examples: $R = \mathbb{C}[\alpha, \beta]$

Principal ideals: If two vertices are connected by an edge, their labels must differ by a multiple of the label on that edge.

(Edge-labels are colored.)

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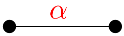

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$$p \bullet \overset{\alpha}{\rule{3cm}{0.4pt}} \bullet \; p + q\alpha$$

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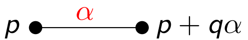

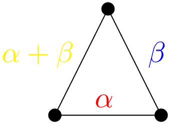

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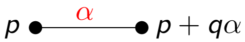

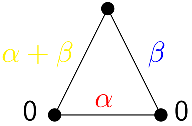

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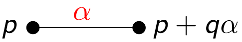

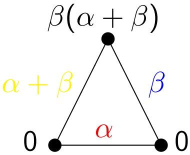

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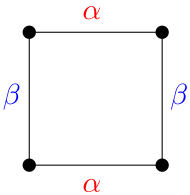

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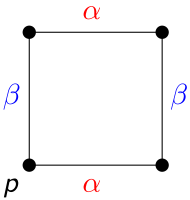

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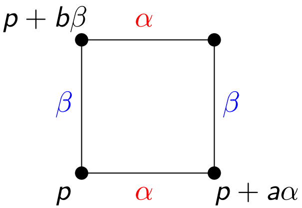

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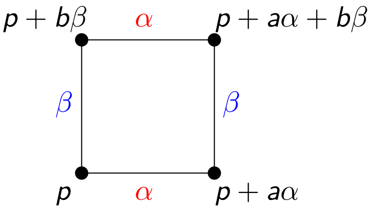
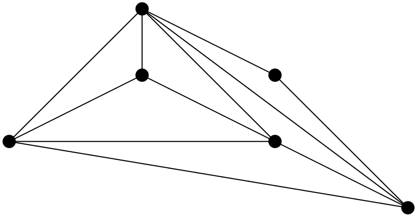


Relation to classical splines: the dual graph

Consider the dual graph Δ^* to a triangulation Δ:

- each triangle becomes a vertex; and
- if two triangles share an edge, draw an edge between the corresponding vertices.

We also label each edge uv in Δ^* by the slope ℓ_{uv} of the corresponding edge in Δ.



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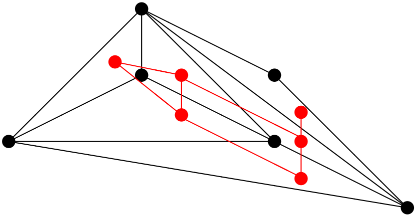

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Classical splines: Given a triangulation Δ of a region in the plane (say), the set of splines is

$$S_d^r(\Delta) = \left\{ \begin{array}{l} \text{piecewise polynomials of degree at most } d \\ \text{that agree on the boundaries with smoothness } r \end{array} \right\}$$

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Theorem (Billera-Rose)

$S_d^r(\Delta)$ is isomorphic to splines over $\mathbb{R}[x_1, \ldots, x_n]/\mathcal{I}_{d+1}$ on Δ^* with edge labels $\alpha(uv) = (\ell_{uv}^r)$.

Background

Splines on lattices

Basic question: Can we explicitly construct a *basis*[1] for the splines on a lattice?

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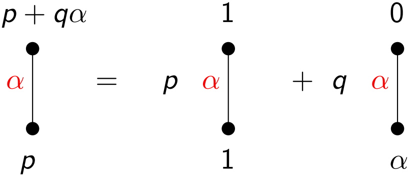

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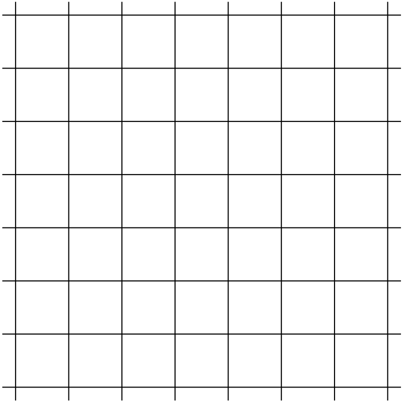

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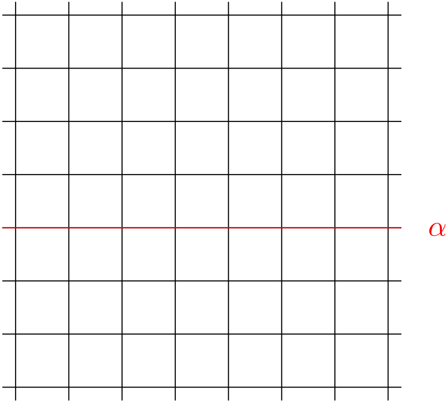

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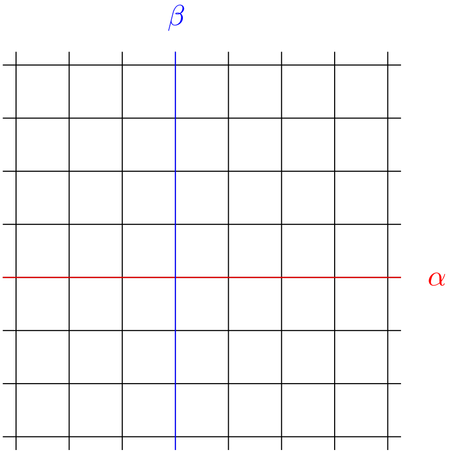

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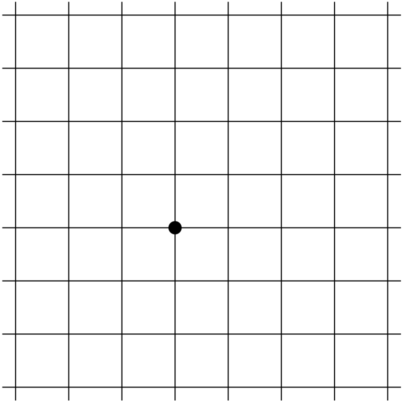

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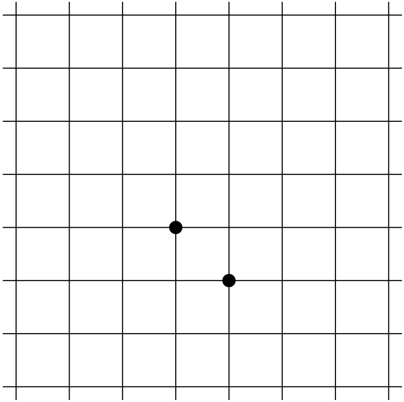

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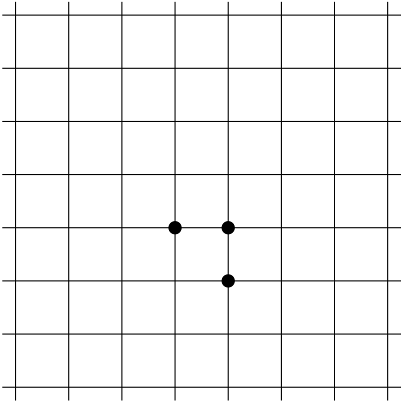

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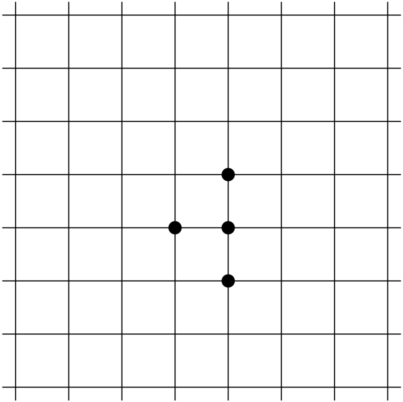

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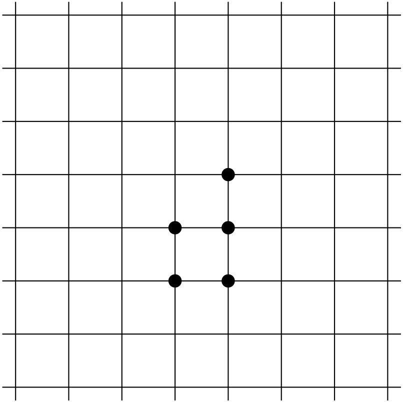

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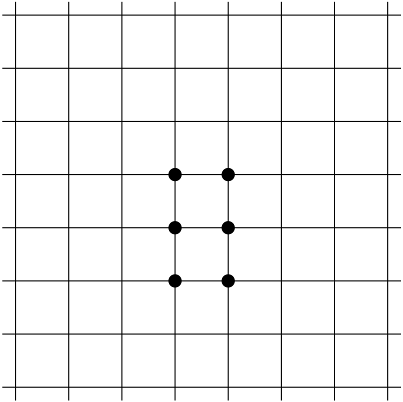

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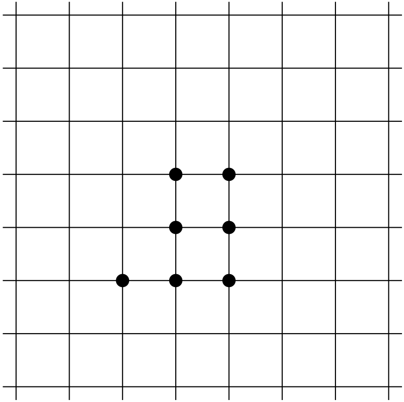

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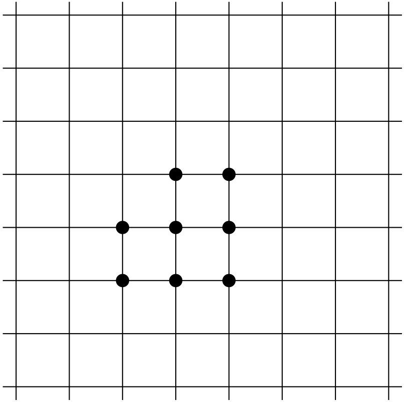

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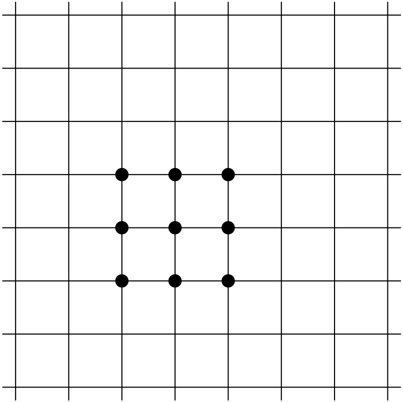

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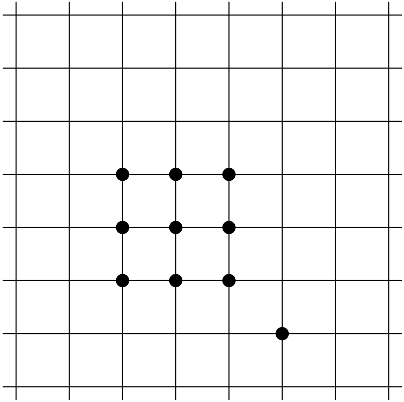
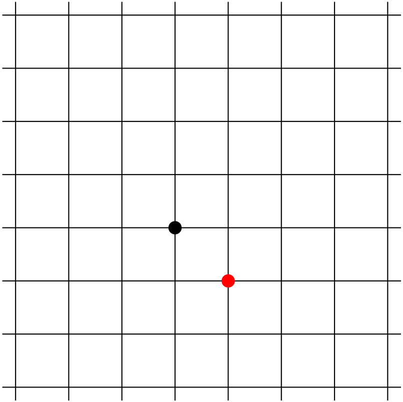


Basis for splines on lattices: support on linear subspaces



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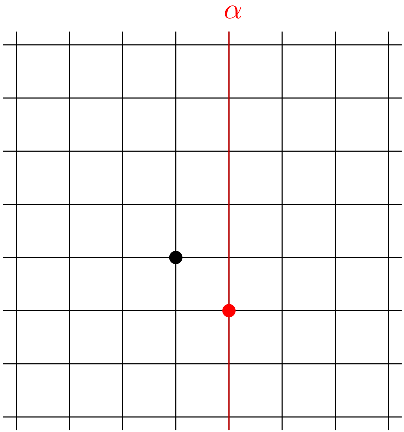

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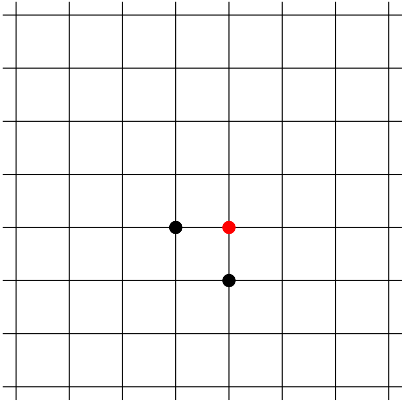

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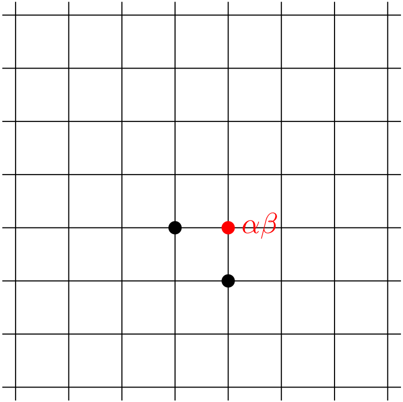

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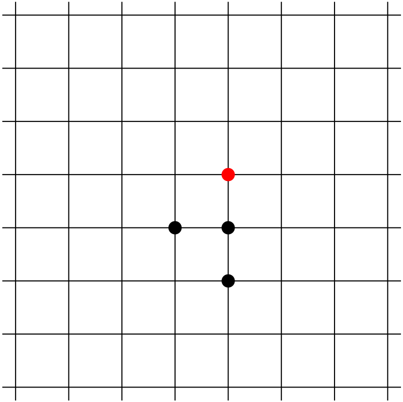

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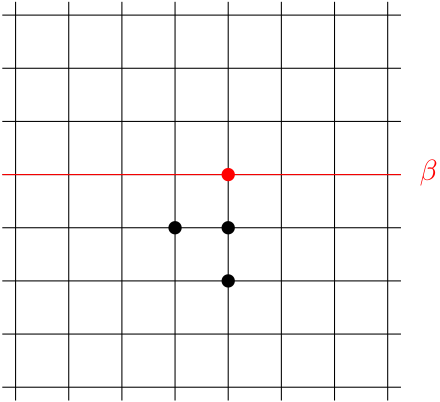

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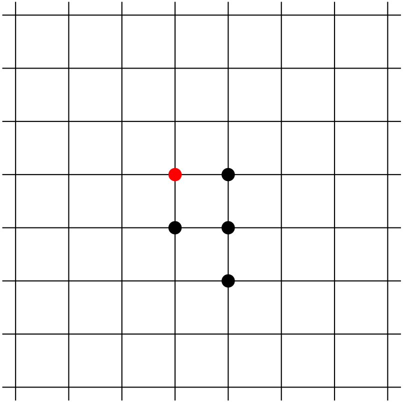

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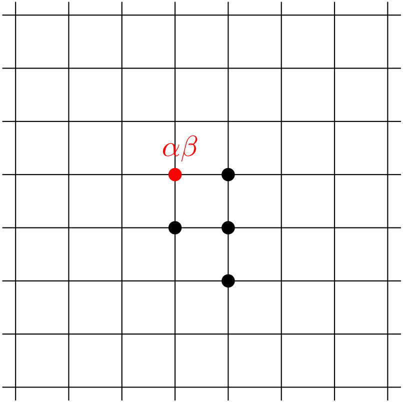

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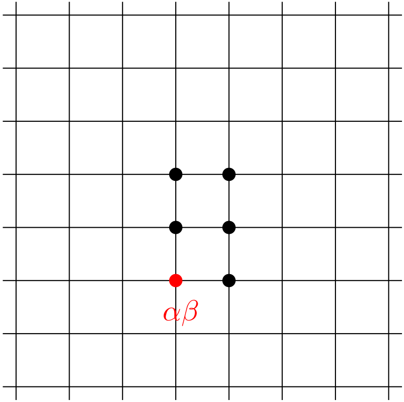

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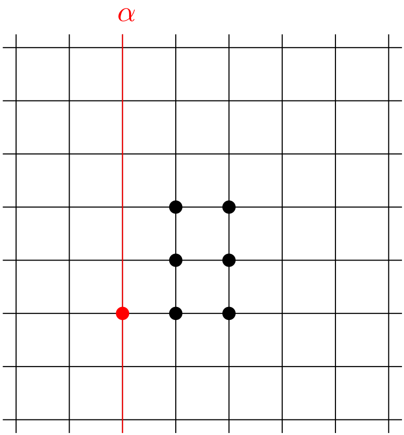

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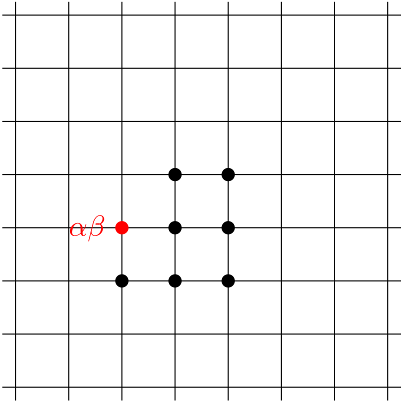

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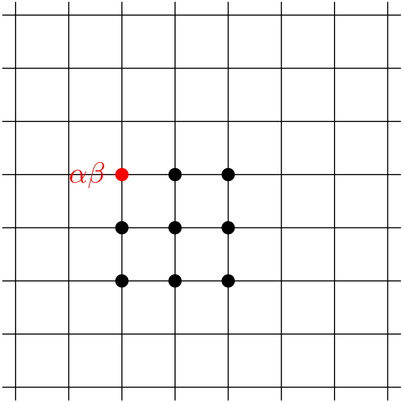
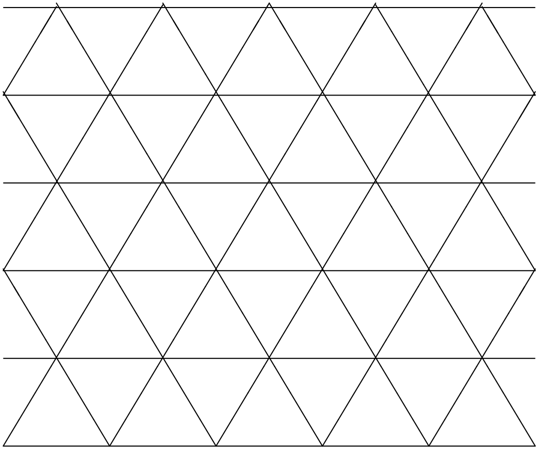


Theorem (T-Mandel-Yun)

This process produces a basis for splines on lattices in $\mathbb{R}^n$.

Root lattices



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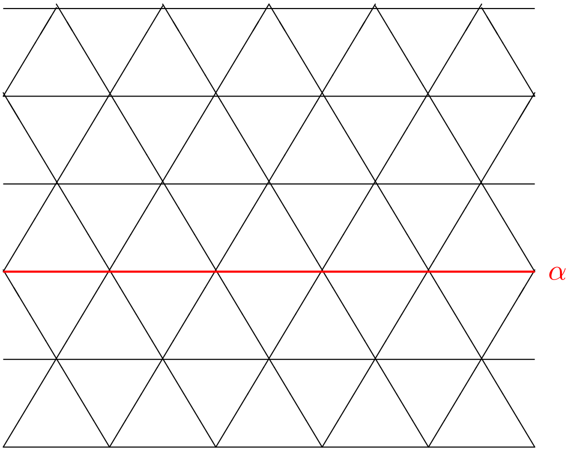

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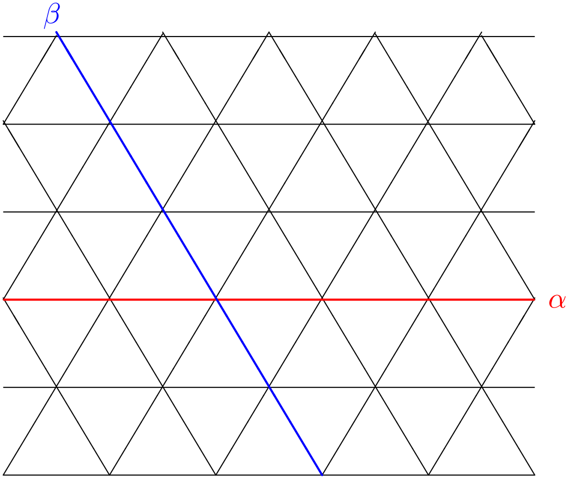

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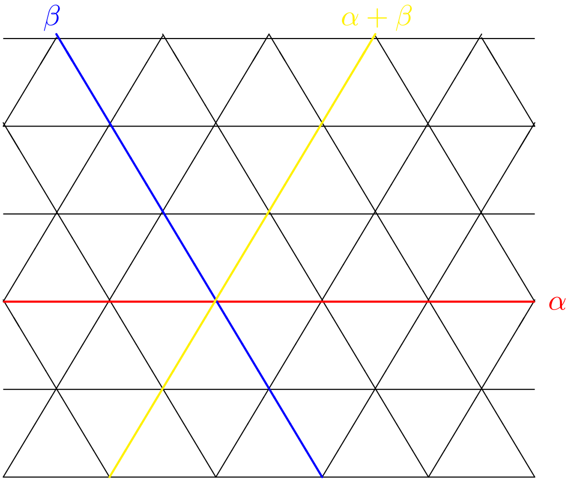

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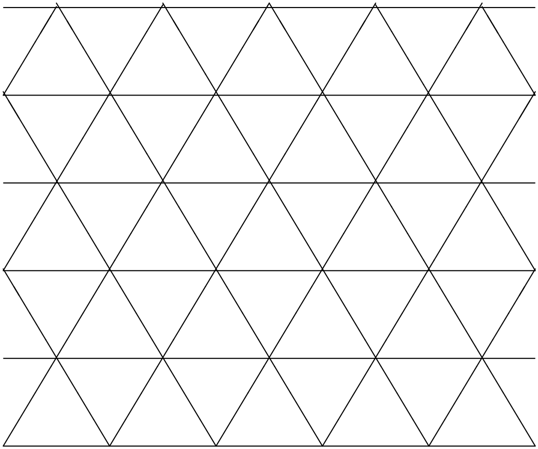

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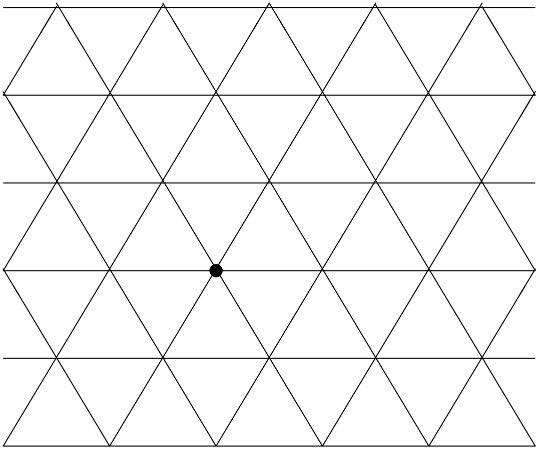

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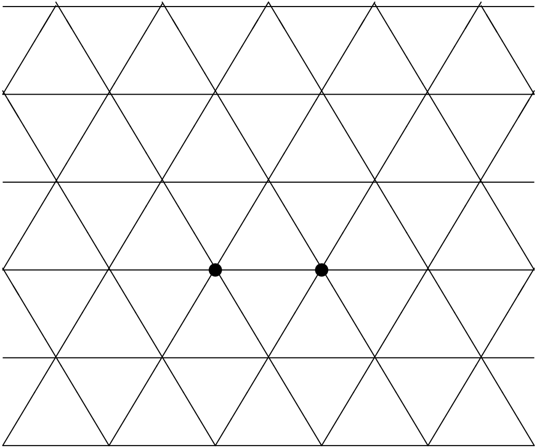

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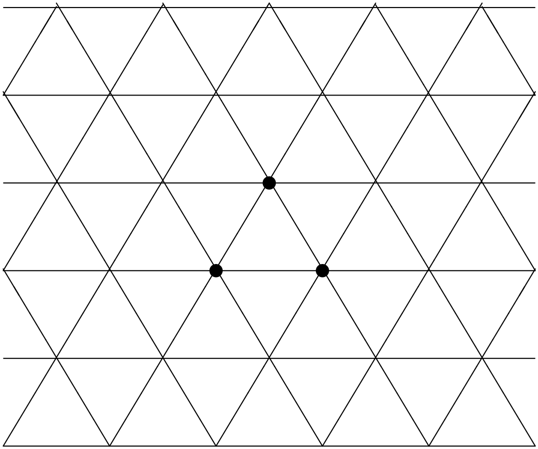

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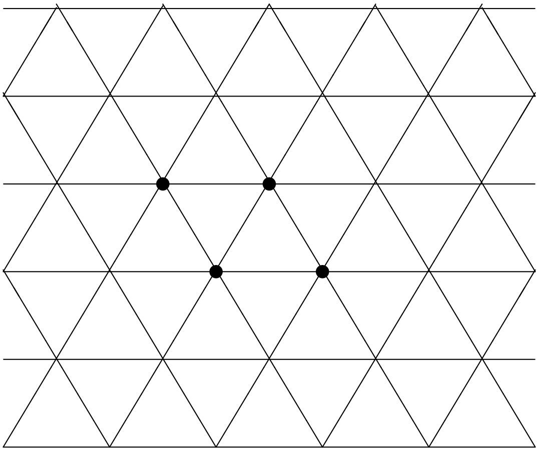

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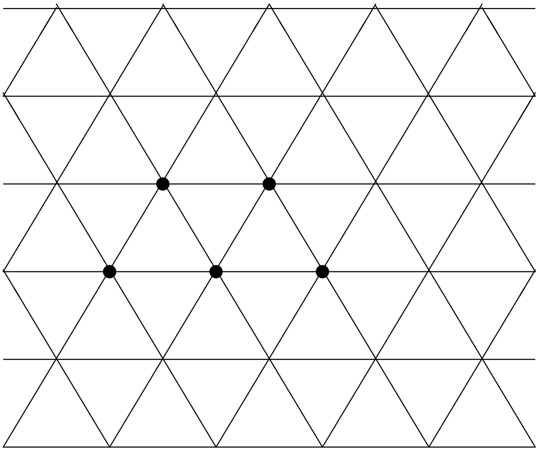

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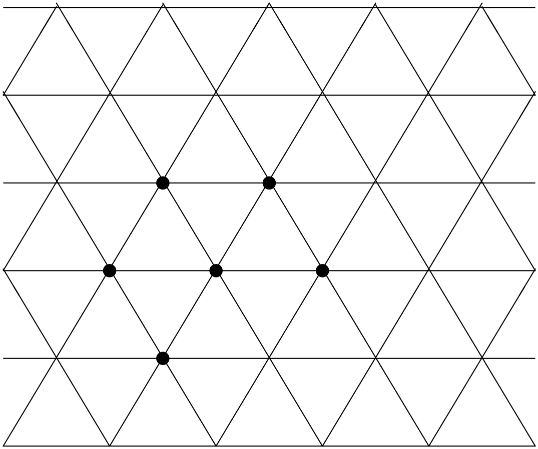

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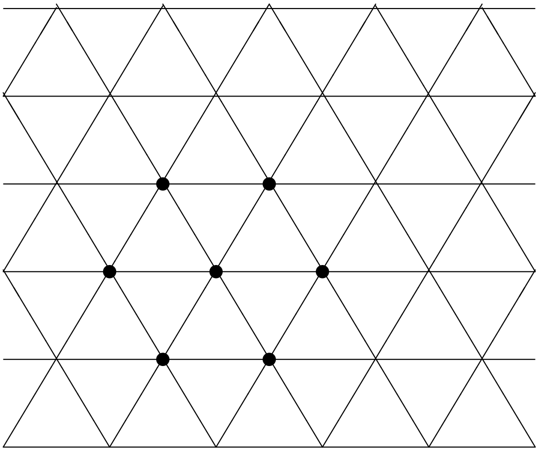

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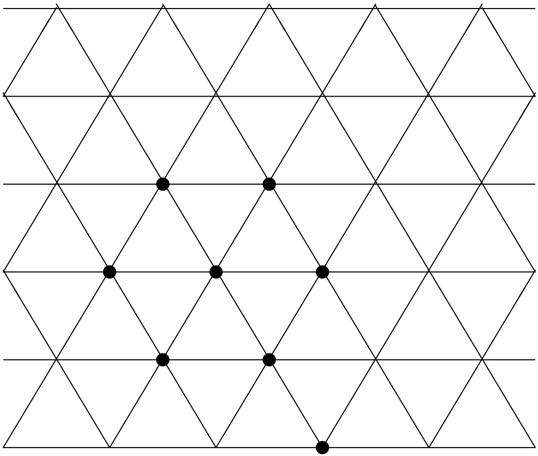

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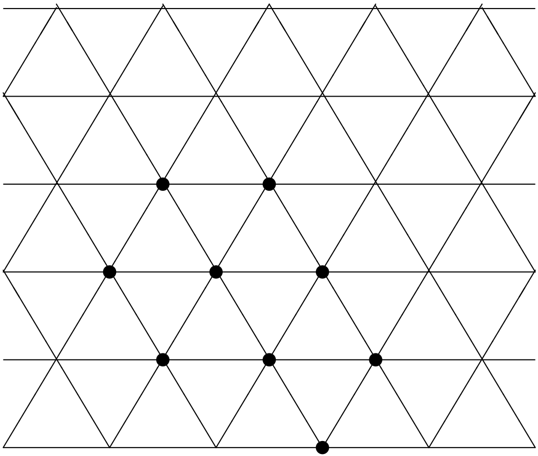

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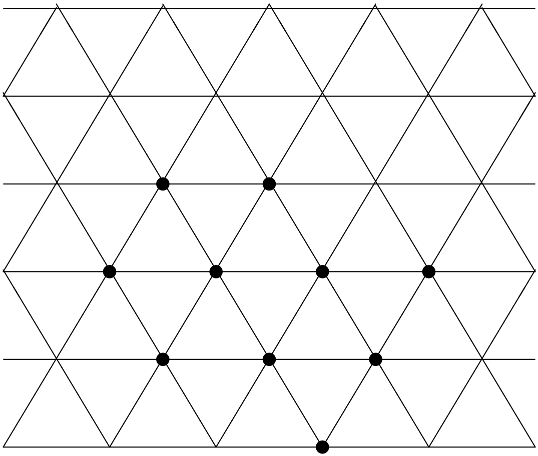

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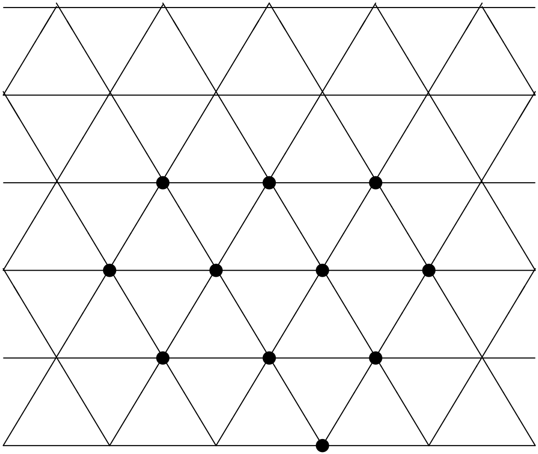

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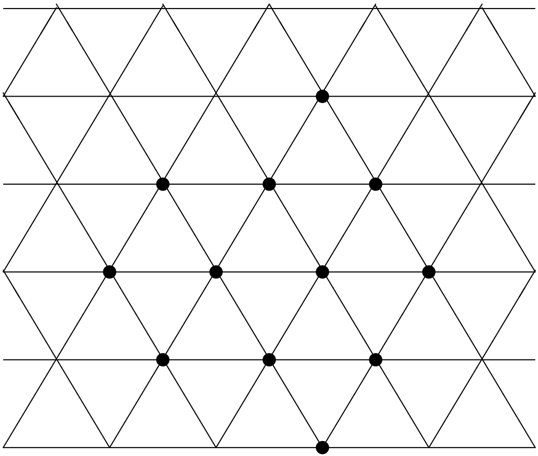

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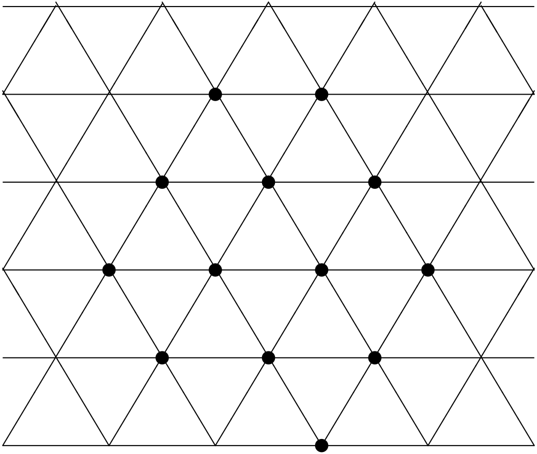

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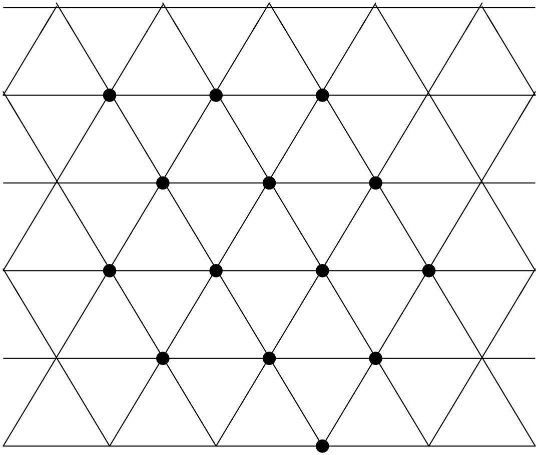

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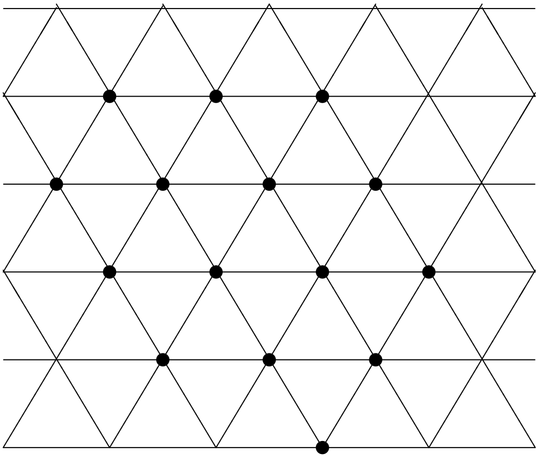

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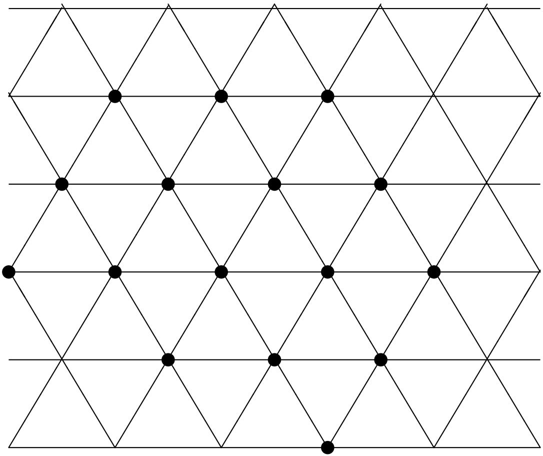
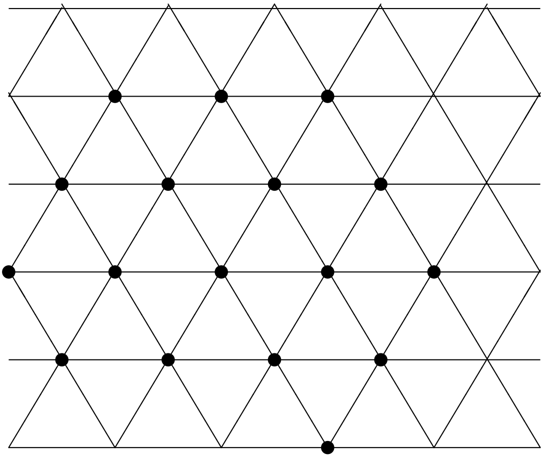
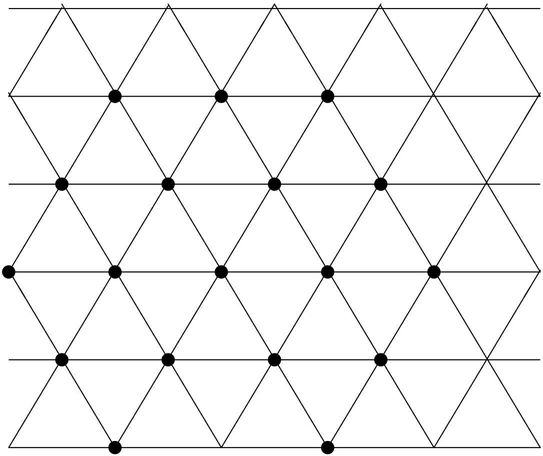


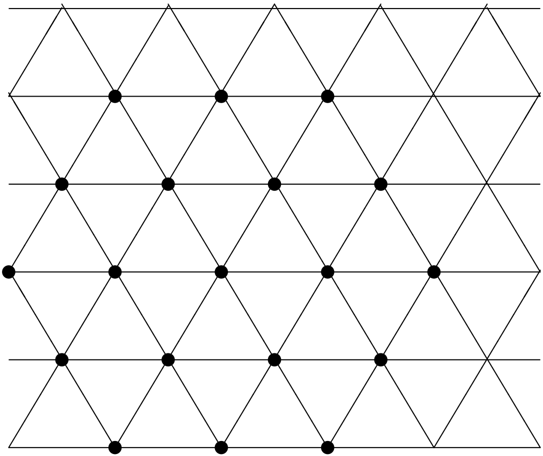
Root lattices: Ordering the vertices



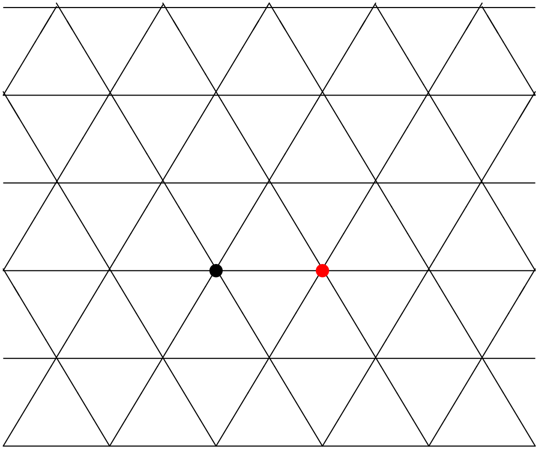
Root lattices: Ordering the vertices



Root lattices: Ordering the vertices



Basis classes



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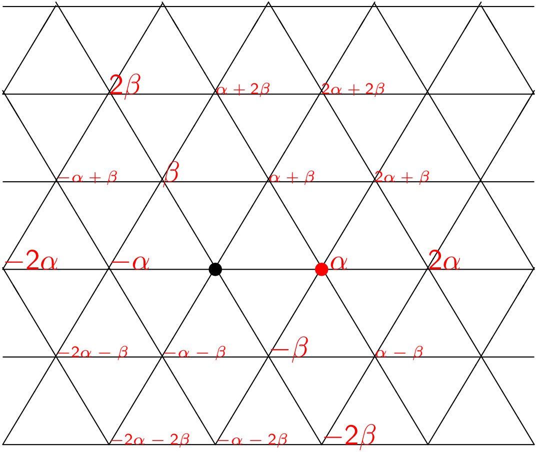

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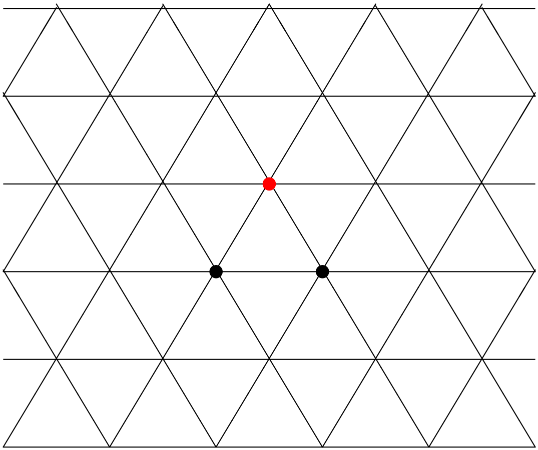

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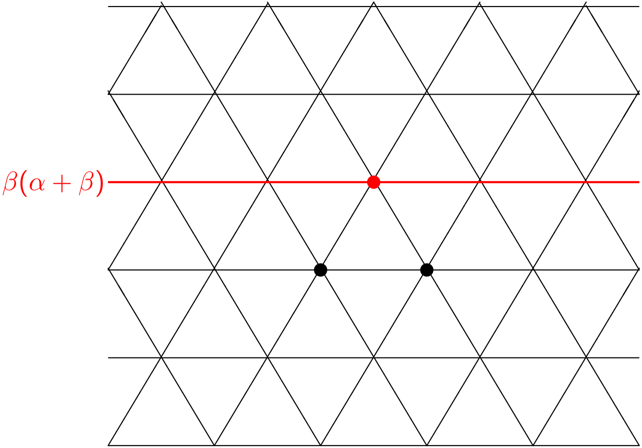

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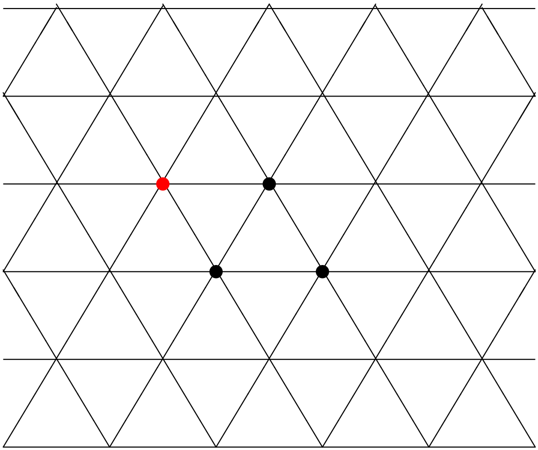

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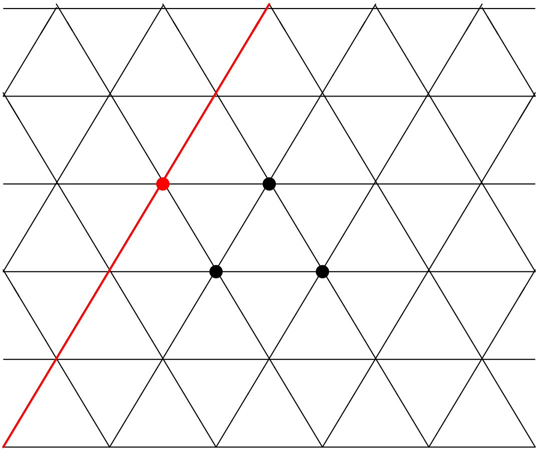
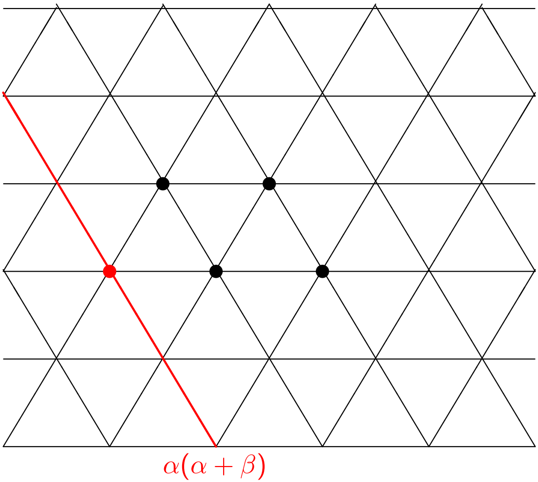


$\alpha\beta$

Basis classes



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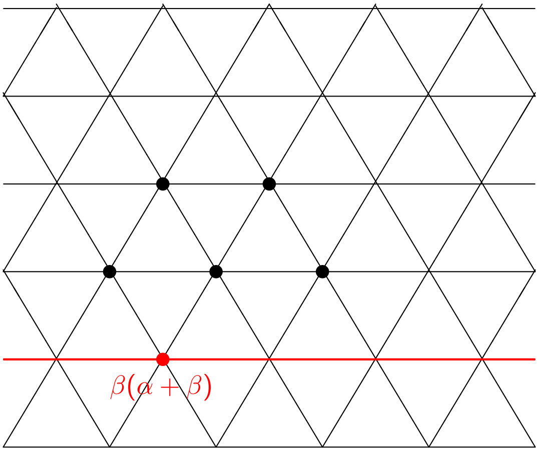

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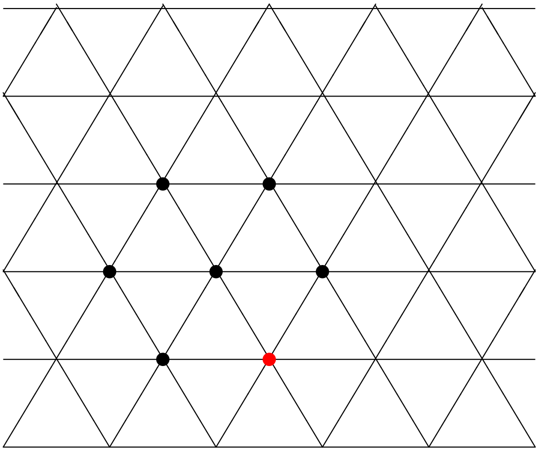

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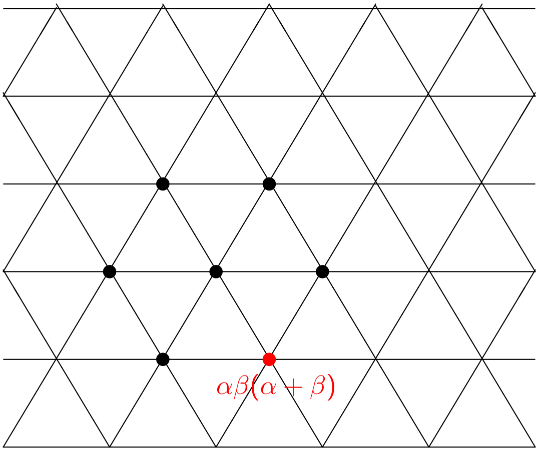
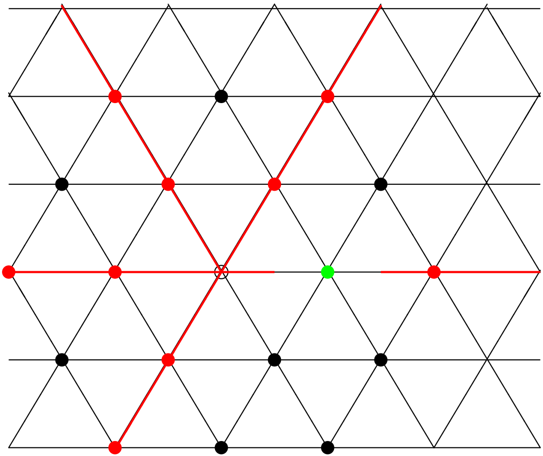


Theorem: [T-Yun] A basis for splines on A_2 root lattice, with dimensions



GKM theory: moment graphs

Suppose X is a “nice” variety with a “good” action of a torus T.

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Then we can create a moment graph:

- T-fixed points become vertices
- 1-dimensional orbits become edges
- label edges with weight of T-action on corresponding orbit

GKM theory: computing equivariant cohomology

Suppose X is an algebraic variety with the action of a torus T.

Theorem

Under certain technical conditions on X and T, the equivariant cohomology $H_T^*(X)$ is isomorphic to the ring of vertex-labelings satisfying the following condition:

For each edge, the labels on the vertices incident to the edge differ by a multiple of the label on the edge.

GKM theory applies to some affine Springer fibers

Theorem

When γ is a regular integral equivalued semisimple element of $\mathfrak{t}$ with weight $k = 1$ then GKM theory applies to the affine Springer fiber of γ.

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When γ is a regular integral equivalued semisimple element of $\mathfrak{t}$ with weight $k = 1$ then GKM theory applies to the affine Springer fiber of γ.

- The proof uses a result of Harada-Henriques-Holm (which says that under appropriate circumstances, GKM theory applies for infinite spaces) and a result of Goresky-Kottwitz-MacPherson (which implies that these X_γ satisfy the necessary conditions).
- Oblomkov-Yun show that the moment graph of these affine Springer fibers is the root lattice.

Punchline

Theorem

The collection of splines on the A_2 root lattice form the equivariant cohomology ring of the affine Springer fiber of γ in $\widetilde{A_2}$ when γ is a regular integral equivalued semisimple element of $\mathfrak{t}$ with weight $k = 1$.

Punchline

Theorem

The collection of splines on the A_2 root lattice form the equivariant cohomology ring of the affine Springer fiber of γ in $\widetilde{A_2}$ when γ is a regular integral equivalued semisimple element of $\mathfrak{t}$ with weight $k = 1$.

Open questions:

- What is a basis for the splines on A_n root lattice for $n > 2$? (This would give the equivariant cohomology ring of a larger family of affine Springer fibers.)
- Can we use the spline construction to say more about group actions on the equivariant cohomology ring?