Interpolating with Hyperplane Arrangements via Generalized Star Configurations Varieties

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Summary

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- Three applications.

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So GSCV's are union of linear subspaces; i.e., *subspace arrangements*.

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3. Let $a = n - \aleph + 1$.

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Observe that in the previous slide and in this example Λ is a set and not a collection, so the linear forms cannot repeat.

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Question: Given $V := \{P_1, \ldots, P_m\}$ distinct points in $\mathbb{P}^{k-1}$, find a linear code such that all its projective codewords of minimum weight are $\phi(P_1), \ldots, \phi(P_m)$.

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Quite messy, but it does the trick.

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We construct, say, p distinct lines $L_1, \ldots, L_p$, and for $k \in \{1, \ldots, p\}$, if each line L_k has $r_k + 1 \geq 2$ points of V on it, then it is considered $r_k \geq 1$ times.

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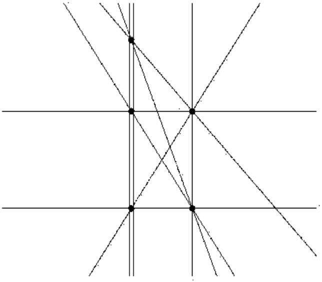

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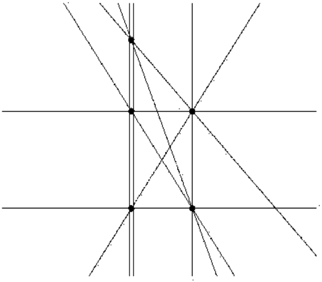

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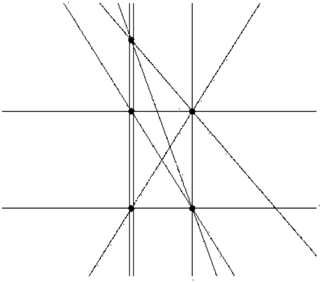

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Dually, $G = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & -1 & -1 & -2 & -2 \end{bmatrix}$, is generating matrix of a linear code with minimum distance $d = 9 - 4 = 5$.

Example 3 (continued).

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There are 5 projective codewords of minimum weight d.

$$\begin{aligned}
\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} G &= \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & -1 & -1 & -2 & -2 \end{bmatrix} \\
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THANK YOU!