Spaces of Splines, Vector Bundles, and Reflexive Sheaves

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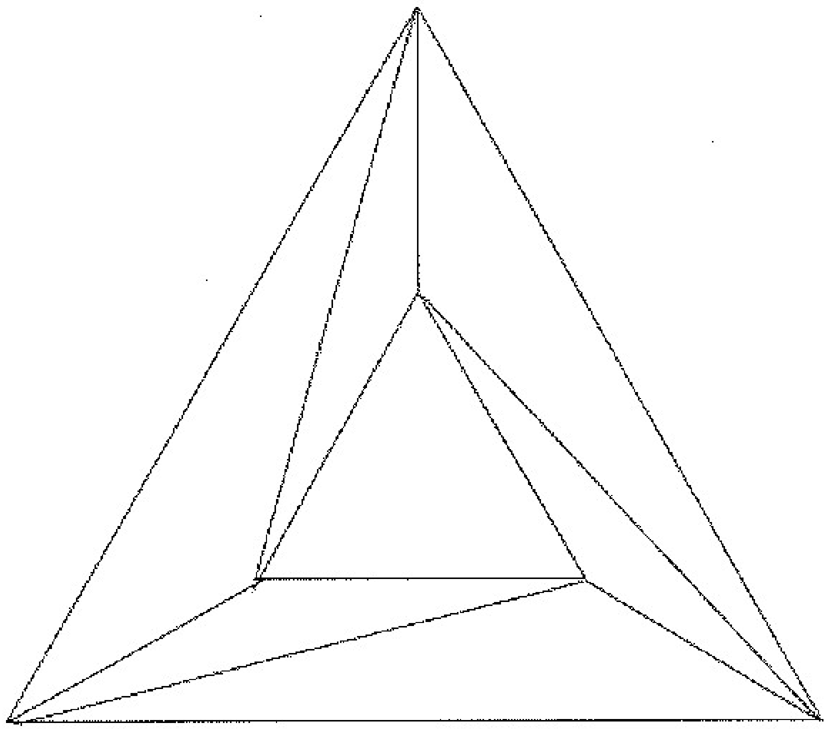
Introduction – The Basic Case

Let $\triangle$ be a triangulation of a simply connected domain $\Omega \subseteq \mathbb{R}^2$ which is homeomorphic to a closed disk.

Generalizations

1) polyhedral subdivision
2) semi-algebraic subdivision
3) $\Omega \subseteq \mathbb{R}^k \quad k > 2$



$S^r_d(\triangle)$=space of piecewise polynomial functions of degree d and smoothness r

So $f \in S^r_d(\triangle)$ if $f|_{\sigma_i} = f_i$ is a polynomial in two variables of degree $\leq d$ on each triangle σ_i in $\triangle$ and f is a C^r-function, i.e. f has continuous derivatives to order r.

The latter means

$$f_i - f_j = g_{ij} l_{ij}^{r+1}$$

where l_{ij} is a linear equation defining the edge $\sigma_i \cap \sigma_j$ when σ_i, σ_j are adjacent.

What can we say about

$$\dim S^r_d(\triangle) = ???$$

Strang's Lower Bound (1973)

$$\dim S^r_d(\triangle) \geq \binom{d+2}{2} + \binom{d-r+1}{2} E_I - \left[\binom{d+2}{2} - \binom{r+2}{2}\right] V_I + \delta$$

where

$$E_I = \# \text{ of interior edges}$$

$$V_I = \# \text{ of interior vertices}$$

and

$$\delta = \sum_{i=1}^{V_I} \sum_{j=1}^{d-r} (r + j + 1 - je_i)_+$$

where e_i is the number of interior edges with different slopes attached to interior vertex v_i.

How is this derived from the standpoint of algebraic geometry?

Using the conformality conditions and the fact that Ω is simply connected, one can produce an exact sequence of vector bundles on $\mathbb{P}^2$

$$(*) \quad 0 \to K(d) \to \bigoplus_{\triangle_1^0} O_{\mathbb{P}^2}(d-r-1) \to \bigoplus_{\triangle_0^0} O_{\mathbb{P}^2}(d)$$

where

$$\dim S_d^r(\triangle) = \binom{d+2}{2} + \dim H^0(\mathbb{P}^2, K(d))$$

We can break $(*)$ into two short exact sequences

$$0 \to K(d) \to \bigoplus_{\triangle_1^0} O_{\mathbb{P}^2}(d-r-1) \to R(d) \to 0$$

$$0 \to R(d) \to \bigoplus_{\triangle_0^0} O_{\mathbb{P}^2}(d) \to C(d) \to 0$$

where C is a skyscraper sheaf supported at the interior vertices so $C(d) = C$.

Using Serre's Vanishing Theorem B, local calculation of C, and the long exact sequences in cohomology, we get for d sufficiently large that $\dim S_d^r(\triangle) =$ Strang's Lower Bound

Schumaker 1979 proved Strang's conjectured lower bound.

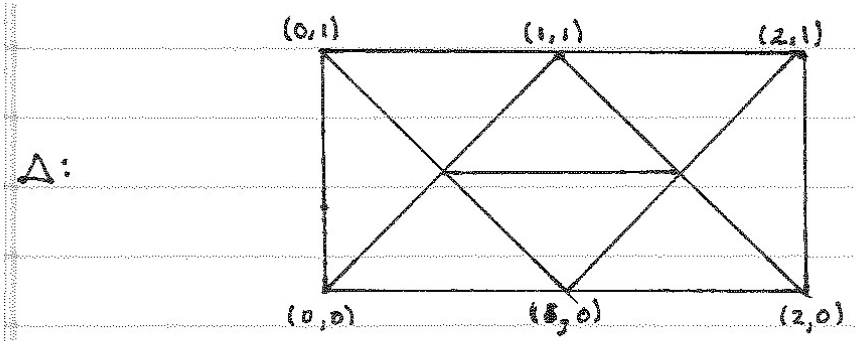
Alfeld and Schumaker 1987: If $d \geq 3r + 2$ the dimension of $S_d^r(\triangle)$ is given by the lower bound.

Alfeld and Schumaker 1990: If $d = 3r + 1$ then for generic $\triangle$ the dimension is given by the lower bound.

For $d = 4$ and $r = 1$ the dimension is given by the lower bound in all cases.

"... As far as I know there has been no real progress on the dimension of $S^r_d(\triangle)$ for general triangulations when $2r+1 \leq d \leq 3r+1$, for many years. In particular, the dimension $S^1_3(\triangle)$, the most interesting case in my opinion, seems to be as inaccessible as ever. ..."
– Peter Alfeld 6/26/17

Example (Stefan Tohaneanu)



$\dim S_3^1(\triangle) = 23 \qquad r = 1 \text{ and } d = 3$

Note: Strang's lower bound in this case is 23

$$\binom{3+2}{2} + \binom{3-1+1}{2} E_I - \left[\binom{3+2}{2} - \binom{1+2}{2} \right] V_I + \delta$$

where

$E_I = 9$ the number of interior edges

$V_I = 2$ the number of interior vertices

$$\delta = \sum_{i=1}^{V_I} \sum_{j=1}^{d-r} (r + j + 1 - je_i)_+$$

e_i is the number of edges with different slopes attached to interior vertex v_i, so $e_i = 3$ for both interior vertices and $\delta = 0$.

But for $S^r_{2r}(\triangle)$ Tohaneanu shows

$$\dim S^r_{2r}(\triangle) > \text{lower bound}$$

In this case the lower bound is

$$\begin{aligned} &4r^2 + \tfrac{9}{2}r + 1 \quad \text{for } r \text{ even} \\ &4r^2 + \tfrac{9}{2}r + \tfrac{1}{2} \quad \text{for } r \text{ odd} \end{aligned}$$

Later Tohaneanu and Minac 2012 showed that in this example, for $d \geq 2r + 1$ the dimension is the lower bound.

Conjecture ("The $2r+1$ Conjecture")

Conjecture

For $d \geq 2r+1$ the dimension of $S_d^r(\triangle)$ is given by the lower bound and this is sharp, i.e. for $d = 2r$ there exist $\triangle$ for which $\dim S_d^r(\triangle)$ exceeds the lower bound.

More algebraic geometry

From the long exact sequence in cohomology for

$$0 \to K(d) \to \bigoplus_{\triangle_1^0} O_{\mathbb{P}^2}(d-r-1) \to R(d) \to 0$$

we get

$$H^1(R(d)) \cong H^2(K(d))$$

but it can be shown (Lau, Stiller) $\dim H^2(K(d)) = 0$ for $d \geq 2r$.

This gives for $d \geq 2r$ and $d > r$.

$$h^0(K(d)) - h^1(K(d)) = f_1^0\binom{d-r+1}{2} - f_0^0\left[\binom{d+2}{2} - \binom{r+2}{2}\right]$$

$$+\sum_{i=1}^{f_0^0}\sum_{j=1}^{d-r}[-e_i j + j + r + 1]_+$$

It follows that for $d \geq 2r$ and $d > r$

$$\dim S_d^r = \text{lower bound}$$

if and only if $H^1(K(d)) = 0$.

So we can re-interpret the $2r + 1$ conjecture as a cohomology vanishing result.

Conjecture 1

For $d \geq 2r + 1$, $\quad H^1(K(d)) = 0.$

In fact one can show if $H^1(K(d_0)) = 0$ for some $d_0 \geq 2r + 1$ then $H^1(K(d)) = 0$ for all $d \geq d_0 \geq 2r + 1$.

Conjecture 2

$H^1(K(2r + 1)) = 0.$

Are there cohomology vanishing theorems that can help us?

Cohomology Vanishing Results

Elenewajg and Forester "Bounding Cohomology Groups of Vector Bundles on $\mathbb{P}^n$," Math. Ann. 246, 251–270 (1980)

Hartshorne, "Stable Vector Bundles," Math. Ann. 238, 229–280 (1978)

A Deeper Look

Notation For $v \in \triangle_0^0$ let

$\epsilon_v =$ number of edges incident to v

$k_v =$ the number of those edges with distinct slopes

$\alpha_v = \lfloor \frac{r+1}{k_v-1} \rfloor$

$K_v =$ bundle associated with splines on the star of v.

Using Schumaker's dimension formula for the star one can show

$$(*)\, K_v \cong O^{s_1}_{\mathbb{P}^2}(-r-1-\alpha_v) \oplus O^{s_2}_{\mathbb{P}^2}(-r-2-\alpha_v)$$
$$\oplus\, O^{s_3}_{\mathbb{P}^2}(-r-1)$$
$$s_1 = (k_v-1)\alpha_v + k_v - r - 2$$
$$s_2 = r + 1 - (k_v-1)\alpha_v$$
$$s_3 = \epsilon_v - k_v \text{ (so } = 0 \text{ for a ``non-singular'' vertex)}$$

Using $(*)$ and

$$0 \to K \to \bigoplus_{v \in \triangle_0^0} K_v \to \mathcal{O}_{\mathbb{P}^2}^{f_1^{00}}(-r-1) \to 0$$

one gets

$$c_1(K) = -f_1^0(r+1)$$

$$c_2(K) = \binom{f_1^0}{2}(r+1)^2 - \binom{r+2}{2} f_0^0 + \frac{1}{2}\sum_{v \in \triangle_0^0}((k_v-1)\alpha_v^2 + (k_v-2r-3)\alpha_v)$$

One can get estimates for $e(K)$, $b(K)$, and $\delta(K)$

Putting these into Elencwajg and Forester's Theorem we get $H^1(K(d)) = 0$ if

$$d \geq c_2(K) - \frac{f_2 - 2}{2(f_2 - 1)}(f_1^0(r+1))^2 + \frac{f_2 - 1}{8}(f_0^0(r+1))^2 + (f_0^0 + 1)(r+1) - 1$$

(Schenck and S. 2001)

Remark

$H^1(K(d))$ for $d \geq 2r+1$ depends only on $H^1(\mathcal{E}(d))$ for a certain 2-bundle $\mathcal{E}$ constructed from K.

First split off line bundle summands of K

$$K \cong \bigoplus_{i=1}^{\ell} O_{\mathbb{P}^2}(a_i) \oplus K_1$$

$$a_1 \geq a_2 \ldots \geq a_\ell$$

Twist K_1 by $O(g)$ so $K(g)$ is generated by global sections (while $K_1(g-1)$ is not). We get a sequence (Serre)

$$o \to O_{\mathbb{P}^2}^{\text{rank } K_1 - 2} \to K_1(g) \to \mathcal{E}(g) \to 0$$

and of course $H^1(K(d)) \cong H^1(\mathcal{E}(d))$.

Questions

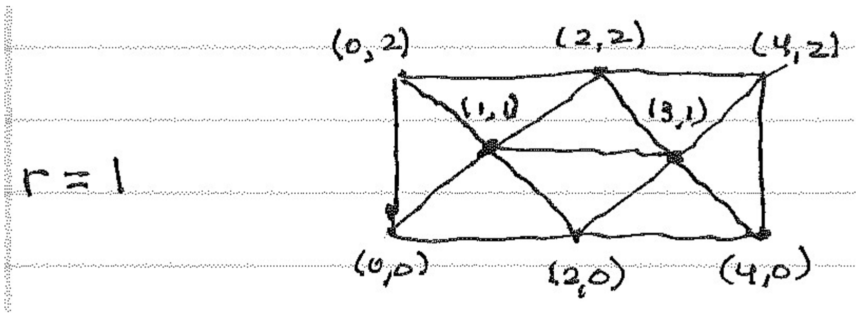
What 2-bundles $\mathcal{E}$ do we get?

Is $\mathcal{E}$ semi-stable / stable?

Can we leverage semi-stability / stability to get better cohomology vanishing estimates?

(see Hartshorne "Stable Reflexive Sheaves")

Example Schenck and Stiller, "Cohomology vanishing and a problem in approximation theory"



For $r = 1$ our estimate using Elencwajg and Forster gives

$$H^1(K(d)) = 0$$

for $d \geq 4$ which is $3r + 1$ (Alfeld and Schumaker).

For $r = 2$

K has rank 7 and $c_1(K) = -27$. Computations show

1) $K \cong O^4_{\mathbb{P}^2}(-3) \oplus K_1 \quad K_1$ indecomposable
2) $K_1|_L \cong O^{\epsilon}_L(-5) \quad L$ generic line in $\mathbb{P}^2$
3) $0 \to O_{\mathbb{P}^2}(-7) \to O^2_{\mathbb{P}^2}(-5) \oplus O^2_{\mathbb{P}^2}(-6) \to K_1 \to 0$ is a resolution of K_1.
4) $\chi(K(d)) = \frac{7}{2}d^2 - \frac{33}{2}d + 21$
5) $c_1(K_1) = -15 \quad c_2(K_1) = 76 \quad b(K_1) = -5$
6) $\delta(K_1) = 1 \quad (\delta = c_2 - \sum_{i<j} b_i b_j)$ where $K_1|_L \cong \oplus O_L(b_i)$ see 2) above.

By (3) or Lau and S., $H^2(K(d)) = 0$ for $d \geq 4$ (which is $2r$).

By Elencwajg and Forester

$$H^1(K(d)) = 0 \text{ for } d \geq \delta - b - 1 = 5$$

(which is $2r + 1$!)

$\dim H^1(K(4)) = 1$ which is Tohaneanu's result.

7) $K_1^{\text{norm}} = K_1(5)$
8) $K_1^{\text{norm}}|_L \cong O_L^3$ so generic splitting type is $(0, 0, 0)$
9) $c_1(K_1^{\text{norm}}) = 0$, $c_2(K_1^{\text{norm}}) = 1$, $\delta(K_1^{\text{norm}}) = 1$, $b(K_1^{\text{norm}}) = 0$ and $H^1(K_1^{\text{norm}}(d)) = 0 \quad d \geq 0$

By results in Elencwajg and Forster $K_1(6)$ is generated by global sections. $K_1(5)$ is not by 3) above. So from Serre we get a sequence

$$0 \to O_{\mathbb{P}^2} \to K_1(6) \to \mathcal{E}(6) \to 0$$

Here $\mathcal{E}(4) = \mathcal{E}^{\text{norm}}$ and

$$c_1(\mathcal{E}^{\text{norm}}) = -1 \text{ and } c_2(\mathcal{E}^{\text{norm}}) = 2$$

Since $c_1(\mathcal{E}^{\text{norm}})$ is odd to show $\mathcal{E}^{\text{norm}}$ stable it suffices to show $H^0(\mathcal{E}^{\text{norm}}) = 0$. But we have

$$0 \to O_{\mathbb{P}^2}(-2) \to K_1(4) \to \mathcal{E}(4) \to 0$$
where $K_1(4) = K_1^{\text{norm}}(-1)$ and $\mathcal{E}(4) = \mathcal{E}^{\text{norm}}$.

This gives $H^0(K_1(4)) \to H^0(\mathcal{E}^{\text{norm}}) \to 0$ and 3) shows $H^0(K_1(4)) = 0$ so $\mathcal{E}$ is stable!

Note for $r = 3$ the $\mathcal{E}$ you get is semi-stable and is the restriction of the null-correlation bundle on $\mathbb{P}^3$ to $\mathbb{P}^2$.

Semistable and Stable Sheaves

Definition

A coherent sheaf F over a complex manifold X is a <u>kth syzergy sheaf</u> if there is an exact sequence

$$0 \to F \to O_X^{\oplus p_1} \to O_X^{\oplus p_2} \to \dots \to O_X^{\oplus p_k}$$

Theorem

The codimension of the singularity set of F (where F_x is not free over $O_{X,x}$) has codimension greater than k.

Let E be a torsion free sheaf on $\mathbb{P}^n$ $\quad n \geq 2$.

Definition

Define $\mu(E) = \frac{c_1(E)}{rkE}$ then E is <u>semi-stable</u> if for a very coherent subsheaf F

$$0 \neq F \subset E$$

we have

$$\mu(F) \leq \mu(E)$$

and <u>stable</u> if for all coherent subsheaves $F \subset E$ with $0 < rkF < rkE$ we have

$$\mu(F) < \mu(E).$$

Fact: For E a vector bundle on $\mathbb{P}^2$ of rank 2 we get E is stable if and only if $H^0(\mathbb{P}^2, E_{\text{norm}}) = 0$. If $c_1(E)$ is even, then E is semistable if and only if $H^0(\mathbb{P}^2, E_{\text{norm}}(-1)) = 0$.

Riemann-Roch for a 2-bundle E over $\mathbb{P}^2$ is

$$\chi(\mathbb{P}^2, E) = \frac{1}{2}(c_1(E)^2 - 2c_2(E) + 3c_1(E) + 4).$$

If E is normalized and semistable, but not stable then $c_1(E) = 0$ and one can show

$$0 \leq h^1(\mathbb{P}^2, E(-1)) - \chi(\mathbb{P}^2, E(-1)) = c_2(E)$$

Theorem

The generic splitting type of a semistable bundle E over $\mathbb{P}^n$, $\underline{a}_E = (a_1, \ldots, a_r) \quad a_1 \geq a_2 \ldots \geq a_r$ has $a_i - a_{i+1} \leq 1$ for all $i = 1, \ldots, r-1$.

So for a normalized semistable 2 bundle E on $\mathbb{P}^n$ you can only get

$(0, 0)$ when $c_1(E) = 0$; or

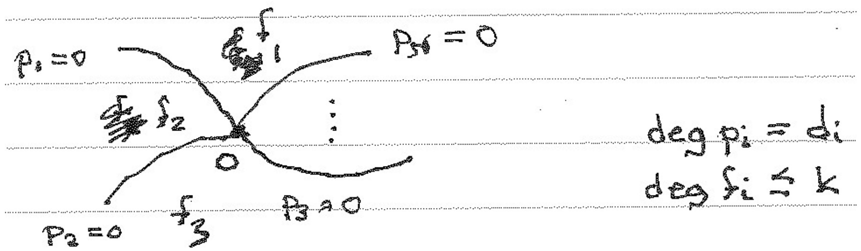
$(0, -1)$ when $c_1(E) = -1$

Semialgebraic Splines

Chui and Wang

(S.) "Certain Reflexive Sheaves on $\mathbb{P}^n_{\mathbb{C}}$ and a Problem in Approximation Theory," Trans. Amer. Math. Soc. 279 (1983), no. 1, 125–142.

DiPasquale, Sottile, and Sun, "Semi-algebraic Splines," preprint.



Conformality Conditions:

$$f_{i+1} - f_i = g_i p_i^{\mu+1} \qquad (f_{N+1} = f_1)$$
$$i = 1, \ldots, N$$

where $deg\; g_i \leq k - d_i(\mu + 1)$. Let S_k^μ = local splines of degree $\leq k$ and smoothness μ

WLOG work in $\mathbb{P}^2$ – homogenize the p_i, g_i, f_i to have degree d_i, $k - d_i(\mu + 1)$, k respectively.

NOTE: By Bezout's Theorem

$$1 \leq \dim H^0(O_x) \leq \min_{i \neq j} d_i d_j (\mu + 1)^2$$

We have exact sequences of coherent sheaves

$$0 \to K(k) \to \bigoplus_{i=1}^{N} O_{\mathbb{P}^2}(k - d_i(\mu+1)) \to \mathcal{I}_x(k) \to 0$$

$$0 \to \mathcal{I}(k) \to O_{\mathbb{P}^2}(k) \to O_x \to 0$$

K is a vector bundle of rank $N - 1$
$\mathcal{I}_x$ is the ideal sheaf of $(p_1^{\mu+1}, ..., p_N^{\mu+1})$
X (zero-dimensional) subscheme in $\mathbb{P}^2$ defined by the $\{p_i^{\mu+1}\}$
O_x the structure sheaf of X – a skyscraper sheaf
supported at the points of X

$$\dim S_k^{\mu} = \dim H^0(K(k)) + \binom{k+2}{2}$$

We see for $k >> 0$

$$\dim H^0(K(k)) = \sum_{i=1}^{N} \binom{k - d_i(\mu+1)+2}{2} - \binom{k+2}{2} + \dim H^0(O_x)$$

So for k sufficiently large

$$\dim S_k^{\mu} = \sum_{i=1}^{N} \binom{k - d_i(\mu+1)+2}{2} + \dim H^0(O_x)$$

Examples:

1. All $d_i = 1$ Chui and Wang computed dim S_k^μ (1981)
   Using this computation one can show (distinct slopes)

$$K(\mu+1+r) \cong \underbrace{\mathcal{O} \oplus \ldots \oplus \mathcal{O}}_{N-1-q} \oplus \underbrace{\mathcal{O}(-1) \oplus \ldots \oplus \mathcal{O}(-1)}_{q}$$

   where we write $\mu + 1 = r(N-1) + q \quad 0 \leq q < N-1.$

2. $N = 3$, $d_i = 2$ for $i = 1, 2, 3 \quad p_1, p_2, p_3$ linearly independent quadrics

$$0 \to K \to \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{I}_x \to 0$$

   K is vector bundle of rank 2 on $\mathbb{P}^2$ with
   $c_1(K) = -6 \quad K_{\text{norm}} = K(3) \quad c_1(K_{\text{norm}}) = 0$

Proposition

K is semistable.

Proof.

Need to show $H^0(K_{\text{norm}}(-1)) = H^0(K(2)) = 0$ but this follow from fact p_1, p_2, p_3 linearly independent. □

If p_1, p_2, p_3 intersect in $s = 1, 2$ or 3 simple points then

a) for $s = 3$ K splits as $O_{\mathbb{P}^2}(-3) \oplus O_{\mathbb{P}^2}(-3)$

b) for $s = 1, 2$ K does not split as $\dim H^1(K(2)) = 3 - s \neq 0$

Using Noether's "AF+BG" Theorem one can show

$$\dim H^0(K(k)) = k^2 - 3k - 1 + s \quad k \geq 3$$

which is the dimension we get for k sufficiently large.

Note: for $s = 1 \quad c_2(K_{\text{norm}}) = c_2(K(3)) = 2$ and for $s = 2 \quad c_2(K_{\text{norm}}) = c_2(K(3)) = 1$

Also $\dim H^0(K(3)) = \dim H^0(K_{\text{norm}}) = s - 1$ which $= 0$ for $s = 1$ so K is stable in this case.

The moduli space $M_{\mathbb{P}^2}(0, 2)$ of stable 2-bundles with $c_1 = 0$, $c_2 = 2$ is a smooth irreducible variety that is well understood.