Algebraic Methods in Spline Theory

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Multivariate Splines and Algebraic Geometry

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- However, non-polynomial functions may be used (Cesare Bracco; numerical analysis)
- Or the cells of the subdivision could be semi-algebraic sets, defined by arbitrary polynomial inequalities (Peter Stiller, Frank Sottile; numerical analysis and algebraic geometry)

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- Dually, the domains could be considered as vertices of a graph (even infinite!) with algebraic gluing condition across edges (Julianna Tymoczko; equivariant cohomology)
- Other work related to splines in this mini: Algebraic geometry and commutative algebra, with applications to interpolation problems (Stefan Tohaneanu, Boris Shekhtman)

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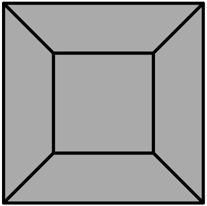

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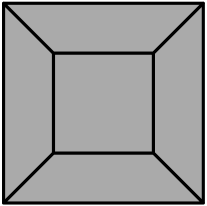

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Notation:

- Δ_i : faces of dimension i (i-faces)
- Δ_i° : interior i-faces
- If $\tau \in \Delta_{n-1}$, $\ell_\tau =$ linear form cutting out affine span of τ

Splines (classical definition, algebraically speaking)

C^r spline on Δ: collection $F = (F_\sigma)$ of polynomials $F_\sigma \in R = \mathbb{R}[x_1, \ldots, x_n]$, for every $\sigma \in \Delta_n$, so that if $\sigma \cap \sigma' = \tau \in \Delta_{n-1}$ then $(\ell_\tau)^{r+1} | (F_\sigma - F_{\sigma'})$.

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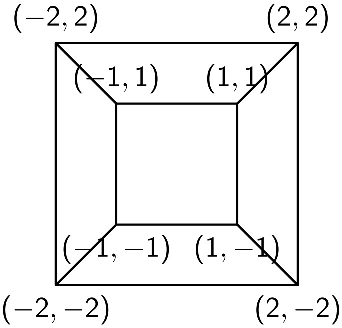

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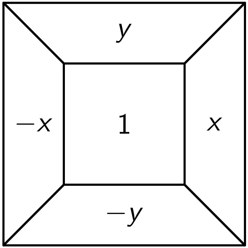

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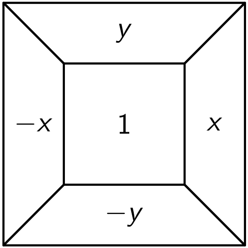

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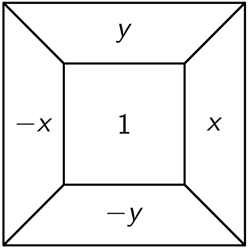

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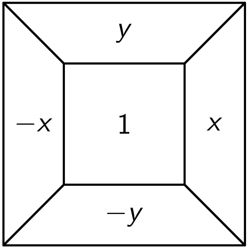

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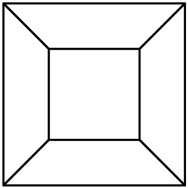
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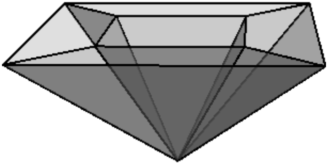
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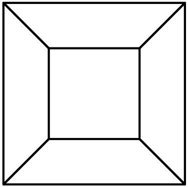

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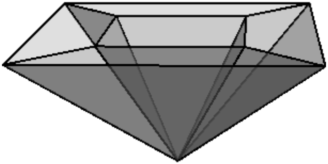

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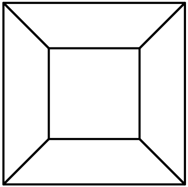

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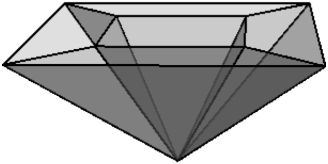

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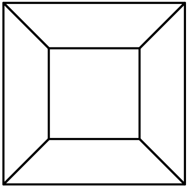

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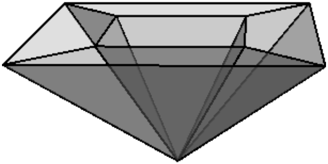

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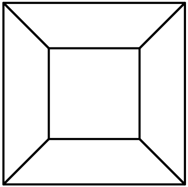

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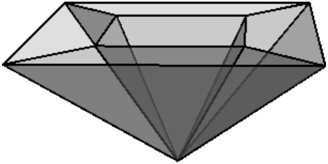

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- Call Δ *central* if $\mathbf{0} \in \sigma$ for every $\sigma \in \Delta_n$

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  - Even $\dim S^1_3(\Delta)$ is unknown! (*generically* given by Schumaker's lower bound (Billera,Whiteley'88))

Freeness questions (Numerical analysis, topology)

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- For (4): analogue of Saito's criterion from arrangement theory identifies when a set of splines forms a free basis for $S^r(\Delta)$

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- $\dim S^0_d(\Delta)$ completely determined by combinatorics of Δ

C^0 non-freeness for polytopal splines

Nonfreeness for Polytopal Complexes [D. '12]

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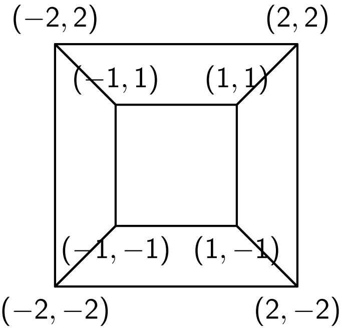

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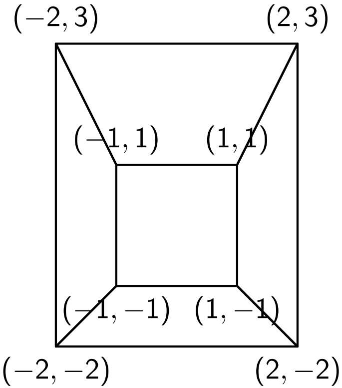

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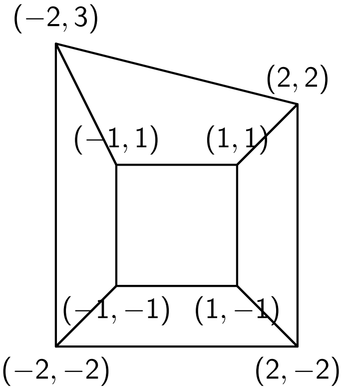

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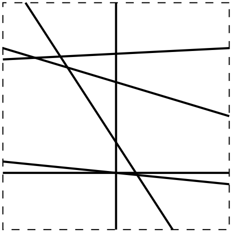

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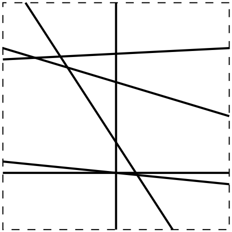

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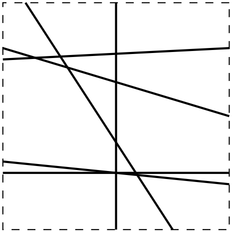

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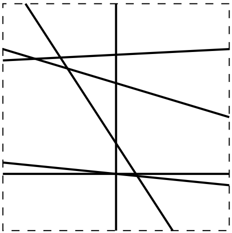

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- $S^r(\widehat{\Delta})$ is also free for any r (Schenck '97)
- Extends to so-called *quasi-crosscut partitions*

Three dimensional crosscut partitions?

$H_1, \ldots, H_k$ linear subspaces of $\mathbb{R}^3$

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Is $\dim S^0(\Delta_{\mathcal{A}})_d$ (or freeness of $S^0(\Delta_{\mathcal{A}})$) combinatorial?

Answer: In general, no.

Example: Ziegler's pair

$\mathcal{A}_t$ = union of hyperplanes defined by the vanishing of the forms (t is considered a parameter):

x	x+y+z	2x+y+z
y	2x+3y+z	2x+3y+4z
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- $\dim S^0(\Delta_0)_1 = \dim S^0(\Delta_t)_1 + 1$ for $t \neq 0$ near zero

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- $\mathcal{A}_t$ is formal except when $t = 0$ (Yuzvinsky '93)

THANK YOU!