Dimensions of Spline Spaces and Commutative Algebra

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Towson University Colloquium

Part I: Background and Central Questions

Piecewise Polynomials

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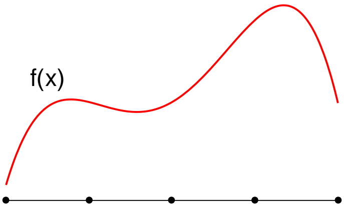

Graph of a function

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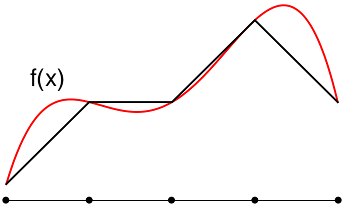

Trapezoid Rule

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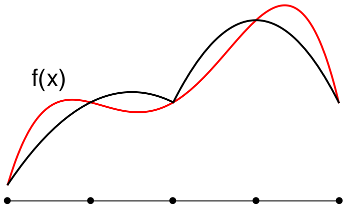

Simpson's Rule

Univariate Splines

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- Subdivide $\Delta = [a, b]$ into subintervals: $\Delta = [a_0, a_1] \cup [a_1, a_2] \cup \cdots \cup [a_{n-1}, a_n]$
- Find a basis for the vector space $C^r_d(\Delta)$ of C^r piecewise polynomial functions on Δ with degree at most d (B-splines!)
- Find best approximation to $f(x)$ in $C^r_d(\Delta)$

Two Subintervals

$\Delta = [a_0, a_1] \cup [a_1, a_2]$ (assume WLOG $a_1 = 0$)

$$
\begin{aligned}
(f_1, f_2) \in C^r_d(\Delta) &\iff f_1^{(i)}(0) = f_2^{(i)}(0) \text{ for } 0 \le i \le r \\
&\iff x^{r+1} | (f_2 - f_1) \\
&\iff (f_2 - f_1) \in \langle x^{r+1} \rangle
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Even more explicitly:

- $f_1(x) = b_0 + b_1 x + \cdots + b_d x^d$
- $f_2(x) = c_0 + c_1 x + \cdots + c_d x^d$
- $(f_0, f_1) \in C^r_d(\Delta) \iff b_0 = c_0, \ldots, b_r = c_r$.

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$$\dim C^r_d(\Delta) = \begin{cases} d+1 & \text{if } d \le r \\ (d+1) + (d-r) & \text{if } d > r \end{cases}$$

Note: $\dim C^r_d(\Delta)$ is polynomial in d for $d > r$.

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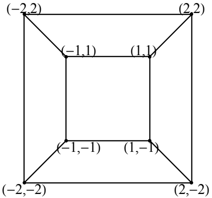

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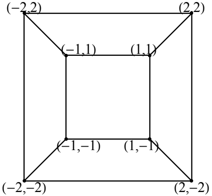

A polytopal complex $\mathcal{Q}$

(Algebraic) Spline Criterion:

- If $\tau \in \Delta_{n-1}$, $l_\tau =$ affine form vanishing on affine span of τ
- Collection $\{F_\sigma\}_{\sigma \in \Delta_n}$ glue to $F \in C^r(\Delta) \iff$ for every pair of adjacent facets $\sigma_1, \sigma_2 \in \Delta_n$ with $\sigma_1 \cap \sigma_2 = \tau \in \Delta_{n-1}$, $l_\tau^{r+1} | (F_{\sigma_1} - F_{\sigma_2})$

The dimension question

Key Fact: $C^r_d(\Delta)$ is a finite dimensional real vector space.

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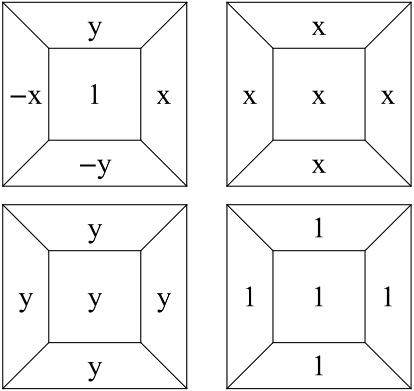
$\dim_{\mathbb{R}} C^0_1(\mathcal{Q}) = 4$



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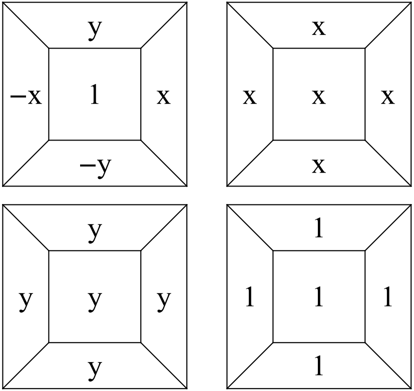


Two central problems in approximation theory:

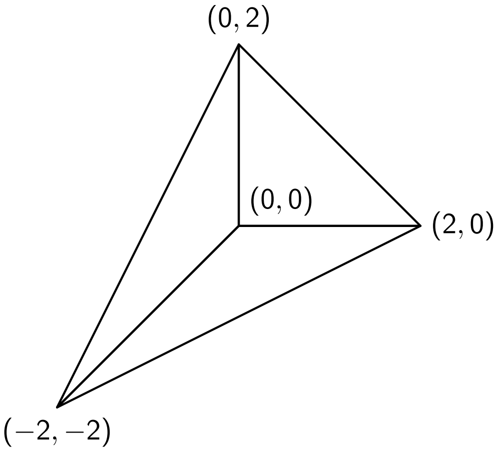
1. Determine $\dim C^r_d(\Delta)$
2. Construct a 'local' basis of $C^r_d(\Delta)$, if possible

Who Cares?

1. Computation of dim $C^r_d(\Delta)$ for higher dimensions initiated by [Strang '75] in connection with finite element method
2. Data fitting in approximation theory
3. Computer Aided Geometric Design (CAGD) - building surfaces by splines [Farin '97]
4. Toric Geometry: Equivariant Chow cohomology rings of toric varieties are rings of continuous splines on the fan (under appropriate conditions) [Payne '06]

Part II: Freeness and (mostly) Continuous Splines

Continuous Splines in Two Dimensions



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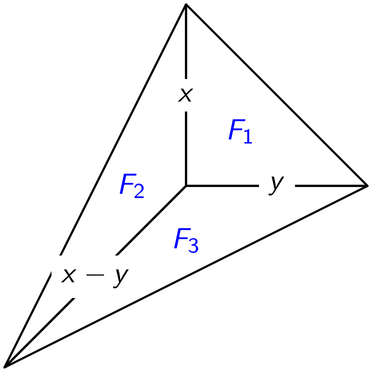

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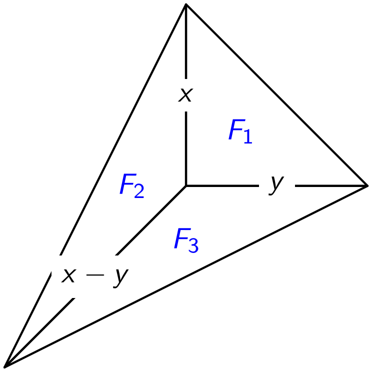
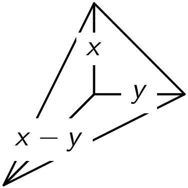


$(F_1, F_2, F_3) \in C^0(\Delta) \iff$
$\exists f_1, f_2, f_3$ so that

$$\begin{aligned} F_1 - F_2 &= f_1 x \\ F_2 - F_3 &= f_2(x-y) \\ F_3 - F_1 &= f_3 y \end{aligned}$$

Freeness



Three splines in $C^0(\Delta)$:

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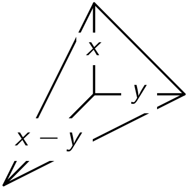

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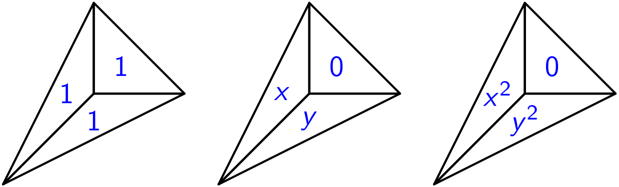

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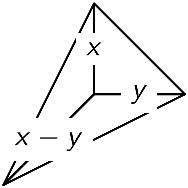

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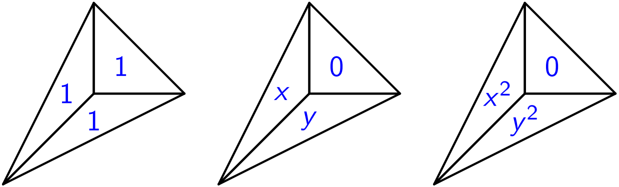

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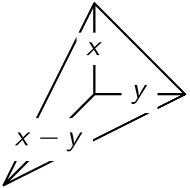

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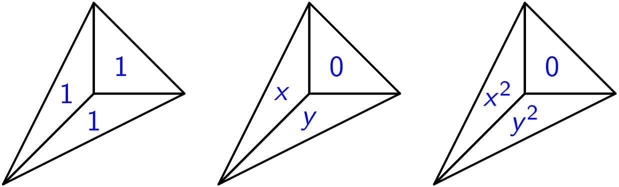

- In fact, every spline $F \in C^0(\Delta)$ can be written uniquely as a polynomial combination of these three splines.
- We say $C^0(\Delta)$ is a **free** $\mathbb{R}[x, y]$-module, generated in **degrees** $0, 1, 2$

Freeness and Dimension Computation

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- $C^0_d(\Delta) \cong \mathbb{R}[x,y]_{\leq d}(1,1,1) \oplus \mathbb{R}[x,y]_{\leq d-1}(0,x,y) \oplus \mathbb{R}[x,y]_{\leq d-2}(0,x^2,y^2)$.
- $\dim C^0_d(\Delta) = \binom{d+2}{2} + \binom{d+1}{2} + \binom{d}{2}$
$= \frac{3}{2}d^2 + \frac{3}{2}d + 1$ for $d \geq 1$

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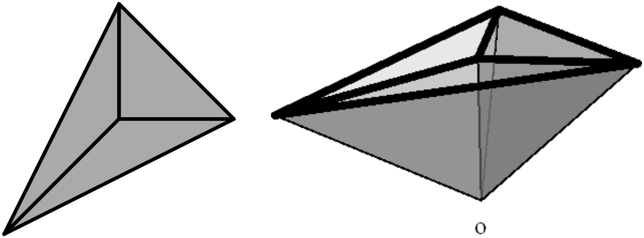
In general, employ a *coning* construction $\Delta \to \widehat{\Delta}$ to homogenize and consider $\dim C^r(\widehat{\Delta})_d$.

Coning Construction

- $\widehat{\Delta} \subset \mathbb{R}^{n+1}$ denotes the cone over $\Delta \subset \mathbb{R}^n$.



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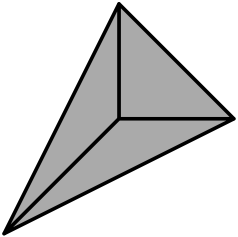
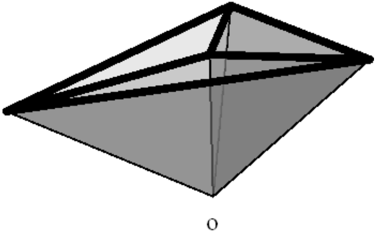


Δ



$\widehat{\Delta}$

- $C^r(\widehat{\Delta})$ is always a **graded** module over $\mathbb{R}[x_0, \ldots, x_n]$
- $C^r_d(\Delta) \cong C^r(\widehat{\Delta})_d$ [Billera-Rose '91]

Consequences of Freeness

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- Many widely-used planar partitions Δ actually satisfy the property that $C^r(\widehat{\Delta})$ is free (type I and II triangulations, cross-cut partitions, rectangular meshes) [Schenck '97]
- Freeness of $C^r(\widehat{\Delta})$ is highly studied:
  - via localization [Billera-Rose '92]
  - via sheaves on posets [Yuzvinsky '92]
  - via dual graphs [Rose '95]
  - via homologies of a chain complex [Schenck '97] (Δ simplicial)

C^0 simplicial splines

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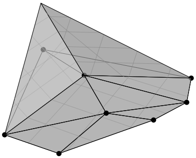

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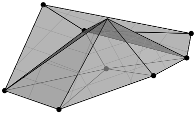

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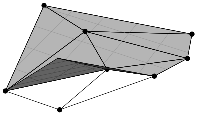

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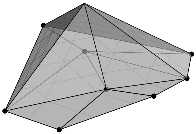

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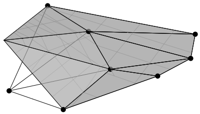

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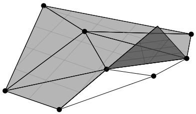

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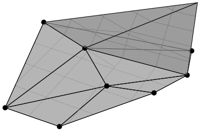

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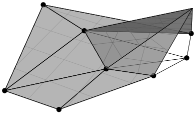

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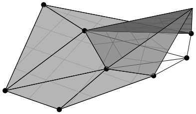

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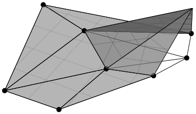

- dim $C^0_1(\Delta)$ = number of vertices of Δ
- $C^0(\Delta)$ is generated as an algebra by tent functions [Billera-Rose '92]

Face rings of simplicial complexes

Face Ring of Δ

Δ a simplicial complex.

$$A_\Delta = \mathbb{R}[x_v | v \text{ a vertex of } \Delta]/I_\Delta,$$

where I_Δ is the ideal generated by monomials corresponding to non-faces.

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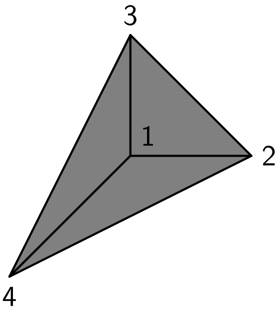

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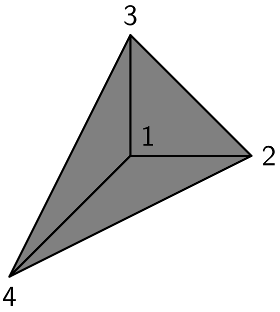

- Nonfaces are $\{1,2,3,4\}, \{2,3,4\}$
- $I_\Delta = \langle x_2x_3x_4 \rangle$
- $A_\Delta = \mathbb{R}[x_1, x_2, x_3, x_4]/I_\Delta$

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C^0 for Simplicial Splines [Billera-Rose '92]

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Why is this an isomorphism?

- Send tent function at vertex v to x_v.
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Consequences:

- $C^0(\widehat{\Delta})$ is entirely combinatorial!
- $\dim C^0_d(\Delta) = \sum_{i=0}^{n} f_i \binom{d-1}{i}$ for $d > 0$, where $f_i = \#i$-faces of Δ.
- If Δ is homeomorphic to a disk, then $C^0(\widehat{\Delta})$ is free as a $S = \mathbb{R}[x_0, \ldots, x_n]$ module.
- If Δ is shellable, then degrees of free generators for $C^0(\widehat{\Delta})$ as S-module can be read off the h-vector of Δ.

Cautionary Tale I

Nonfreeness for Polytopal Complexes [D. '12]

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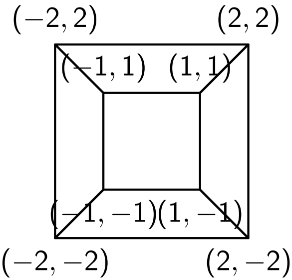

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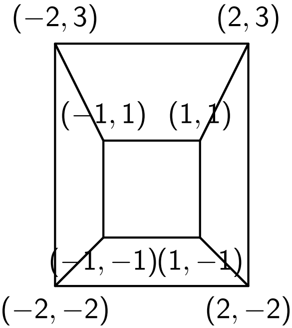

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Nonfreeness for Polytopal Complexes [D. '12]

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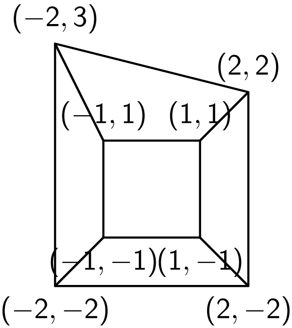

$C^0(\widehat{\Delta})$ is a **free** $\mathbb{R}[x,y,z]$-module

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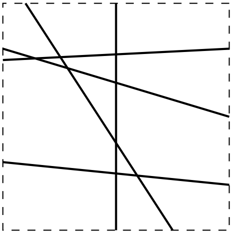

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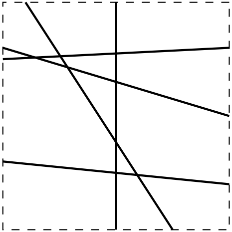

- Basis for $C^r_d(\Delta)$ and $\dim C^r_d(\Delta)$ [Chui-Wang '83]
- $C^r(\widehat{\Delta})$ is free for any r [Schenck '97]

Cautionary Tale II: Ziegler's Pair

Cross-cut partitions fail to be free in $\mathbb{R}^3$!

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$\mathcal{A}_t =$ union of hyperplanes defined by the vanishing of the forms (t is considered a parameter):

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- Let Δ_t be the polytopal complex formed by closures of connected components of $[-1,1] \times [-1,1] \times [-1,1] \setminus \mathcal{A}_t$. (there are 62 polytopes)

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- Let Δ_t be the polytopal complex formed by closures of connected components of $[-1,1] \times [-1,1] \times [-1,1] \setminus \mathcal{A}_t$. (there are 62 polytopes)
- $C^0(\Delta_t)$ is free if and only if $t \neq -5, 0$!

Part III: How Big is Big Enough?

The Hilbert polynomial

From commutative algebra

- $\dim C^r_d(\Delta) = \dim C^r(\widehat{\Delta})_d$ is a polynomial in d for $d \gg 0$
- This is the *Hilbert polynomial* of $C^r(\widehat{\Delta})$, denoted $HP(C^r(\widehat{\Delta}), d)$

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Main questions:

- What is a formula for $HP(C^r(\widehat{\Delta}), d)$?
- How large must d be so that $\dim C^r_d(\Delta) = HP(C^r(\widehat{\Delta}), d)$?

Different approaches for computing $\dim C^r_d(\Delta)$

Analytic Techniques:

- Find upper and lower bounds for $\dim C^r_d(\Delta)$ by explicitly representing polynomials on each polygon and deriving rank conditions on coefficients
- For triangulations, upper and lower bounds agree for $d \geq 3r+1$ [Alfeld-Schumaker '90]

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Algebraic Techniques:

- Find the polynomial $HP(C^r(\widehat{\Delta}), d)$ using Euler characteristic of the Billera-Schenck-Stillman chain complex $\mathcal{R}/\mathcal{J}$ [Billera '89, Schenck-Stillman '97]
- Find when dim $C^r_d(\Delta) = HP(C^r(\widehat{\Delta}), d)$ by
  - Analyzing homologies of $\mathcal{R}/\mathcal{J}$ (done for triangulations in [Mourrain-Villamizar '13])
  - Bounding *regularity* of $C^r(\widehat{\Delta})$ [Schenck-Stiller '02, D. '16]

Planar simplicial splines of large degree

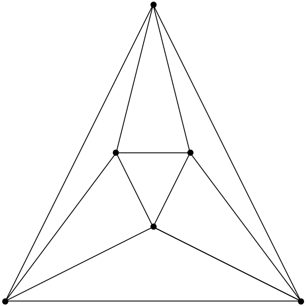
Planar Simplicial Dimension [Alfeld-Schumaker '90]

If $\Delta \subset \mathbb{R}^2$ is a simply connected triangulation and $d \geq 3r+1$,

$$\dim C^r_d(\Delta) = \binom{d+2}{2} + \binom{d-r+1}{2} f_1^0 - \left(\binom{d+2}{2} - \binom{r+2}{2}\right) f_0^0 + \sigma,$$

- f_i^0 is the number of interior i-dimensional faces.
- $\sigma = \sum \sigma_i$.
- $\sigma_i = \sum_j \max\{(r+1+j(1-n(v_i))), 0\}$.
- $n(v_i) = \#$ distinct slopes at an interior vertex v_i.

Morgan-Scot triangulation



$$\dim C_2^1(\mathcal{T}) = 7$$

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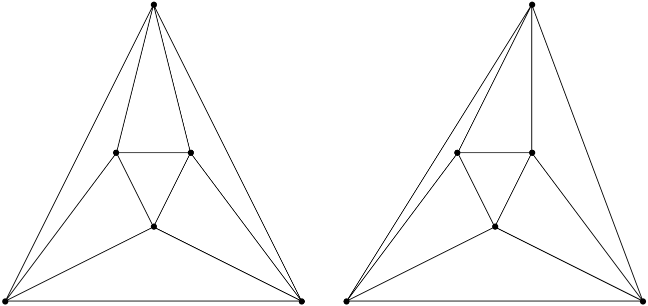

$\dim C^1_2(\mathcal{T}) = 7$ $\qquad$ $\dim C^1_2(\mathcal{T}') = 6$

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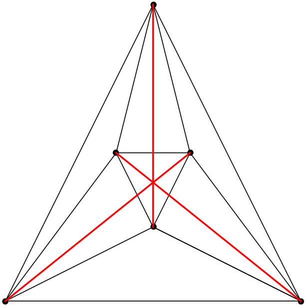

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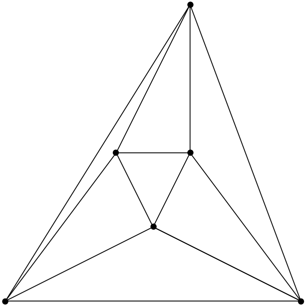

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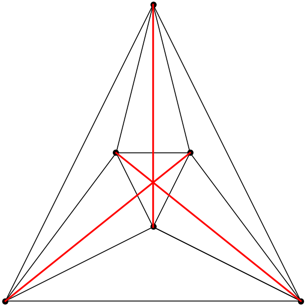

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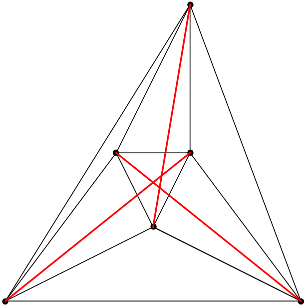

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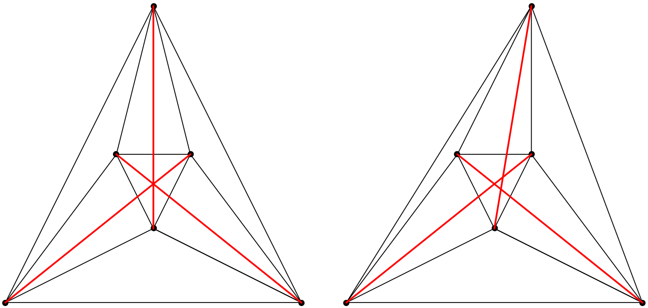

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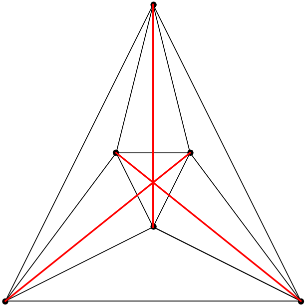

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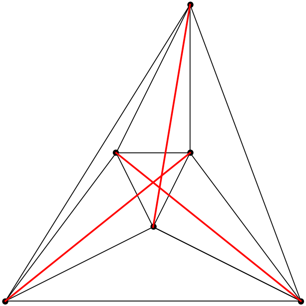

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Conjecture [Schenck]

Alfeld-Schumaker formula for $\dim C^r_d(\Delta)$ holds for $d \geq 2r + 1$.

Planar non-simplicial splines of large degree

Planar non-simplicial dimension [McDonald-Schenck '09]

If $\Delta \subset \mathbb{R}^2$ is a simply connected polytopal complex,

$$\dim C^r_d(\Delta) = f_2\binom{d+2}{2} + f_1^0\left(\binom{d+2}{2} - \binom{d-r+1}{2}\right) - \sigma,$$

- f_i^0 is the number of interior i-dimensional faces.
- $\sigma_i =$ contribution from vertices of Δ (and possibly some non-vertices!)
- $\sigma = \sum \sigma_i$

Bad behavior in non-simplicial case

Disagreement in high degree [D. '14]

If $\Delta \subset \mathbb{R}^2$ is not simplicial, may have $\dim C^r_d(\Delta) \neq HP(C^r(\widehat{\Delta}), d)$ for d as high as $(F-1)(r+1)-2$, where F is maximum number of edges in the boundary of a 2-cell.

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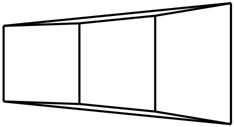

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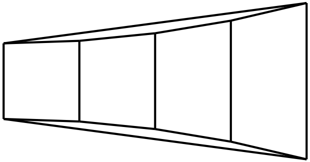

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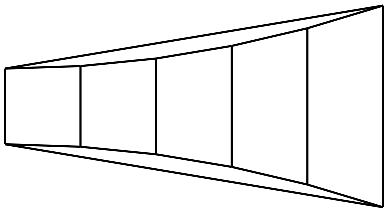

$$\dim C^0_d(\widehat{\Delta}) = \frac{7}{2}d^2 - \frac{7}{2}d + 1 \text{ for } d \geq 4$$

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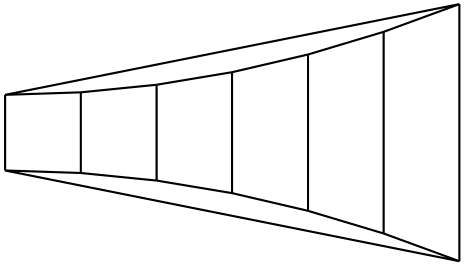

$$\dim C^0_d(\widehat{\Delta}) = \frac{8}{2}d^2 - \frac{10}{2}d + 1 \text{ for } d \geq 5$$

Agreement for non-simplicial splines

Theorem: Using McDonald-Schenck Formula [D. '16]

$\Delta \subset \mathbb{R}^2$ a planar polytopal complex. Let $F =$ maximum number of edges appearing in a polytope of Δ. Then $\dim C^r_d(\Delta) = HP(C^r(\widehat{\Delta}), d)$ for $d \geq (2F-1)(r+1)-1$.

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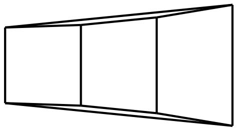

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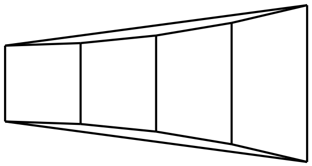

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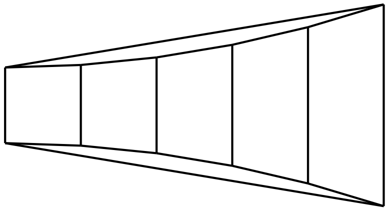

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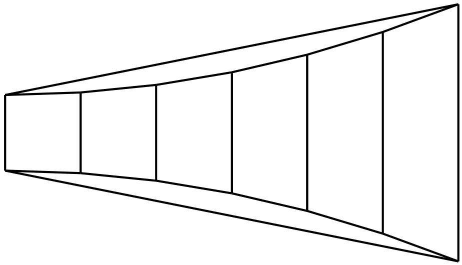

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Part IV: Semi-algebraic Splines

Curved Partitions

More general problem: Compute $\dim C^r_d(\Delta)$ where Δ is a partition whose arcs consist of irreducible algebraic curves.

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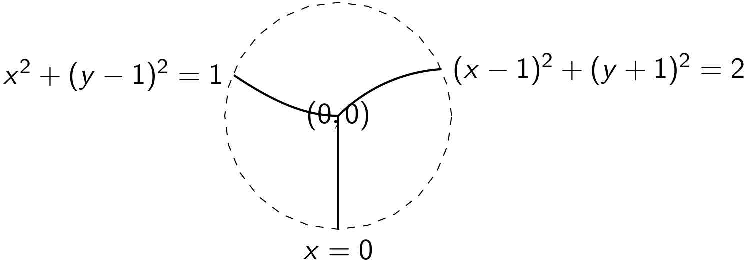

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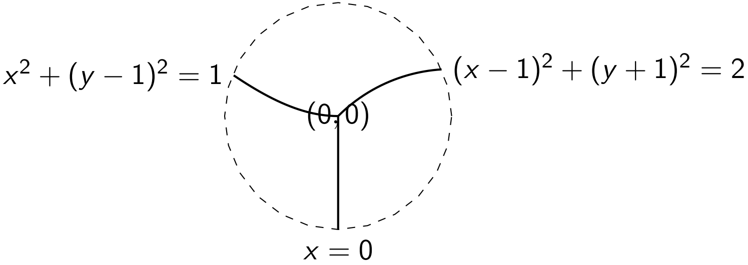

Call functions in $C^r(\Delta)$ *semi-algebraic splines* since they are defined over regions given by polynomial inequalities, or semi-algebraic sets.

Semi-algebraic Splines

Work in semi-algebraic splines:

- First definitions made in [Wang '75] - algebraic criterion for splines carries over verbatim

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- Studied using sheaf-theoretic techniques [Stiller '83]
- Recent work suggests semi-algebraic splines may be increasingly useful in finite element method [Davydov-Kostin-Saeed '16]

Linearizing

- Focus on $\Delta \subset \mathbb{R}^2$ with single interior vertex at $(0,0)$.
- Let Δ_L be the subdivision formed by replacing curves by tangent rays at origin

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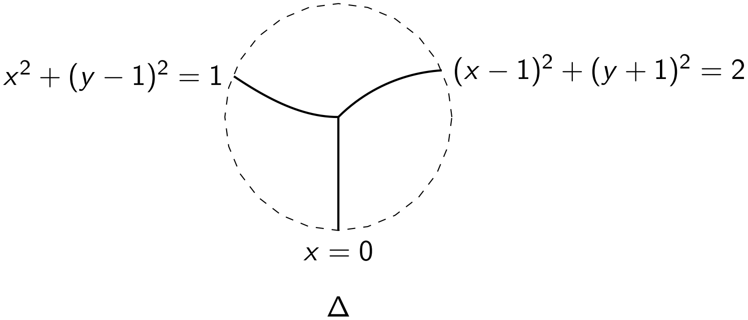

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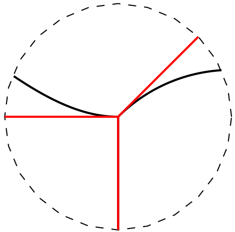

Tangent Lines

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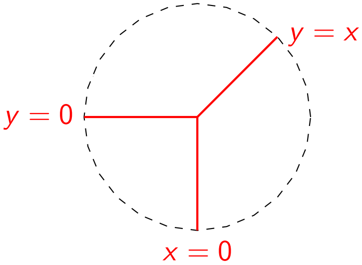

Δ_L

Linearizing

Theorem: Linearizing $\dim C^r_d(\Delta)$ [D.-Sottile-Sun '16]

Let Δ consist of n irreducible curves of degree $d_1, \ldots, d_n$ meeting at $(0,0)$ with distinct tangents and no common zero in $\mathbb{P}^2(\mathbb{C})$ other than $(0,0)$. Then, for $d \gg 0$,

$$\dim C^r_d(\Delta) = \dim C^r_d(\Delta_L) + \sum_{i=1}^{n} \left(\binom{d+2-d_i(r+1)}{2} - \binom{d-r-1}{2} \right)$$

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- Proof uses saturation and toric degenerations (from commutative algebra)
- Bounds on d for when equality holds are also considered, using regularity

Part V: Open Questions

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- More generally (planar polytopal complexes): Compute $\dim C^r_d(\Delta)$ for $r+1 \leq d \leq (2F-1)(r+1)$ (F maximum number of edges in a two-cell)

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- More generally (planar triangulations): Compute $\dim C^r_d(\Delta)$ for $r+1 \leq d \leq 3r+1$
- More generally (planar polytopal complexes): Compute $\dim C^r_d(\Delta)$ for $r+1 \leq d \leq (2F-1)(r+1)$ (F maximum number of edges in a two-cell)
- If $\Delta \subset \mathbb{R}^3$, $\dim C^r_d(\Delta)$ is not known for $d \gg 0$ except for $r=1$, $d \geq 8$ on generic triangulations [Alfeld-Schumaker-Whitely '93]. (connects to unsolved problem in algebraic geometry - the Segre-Harbourne-Gimigliano-Hirschowitz conjecture)

Open Questions

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- Characterize freeness $C^r(\Delta)$. Start with C^0 splines on cross-cut partitions Δ in $\mathbb{R}^3$.
- Compute dim $C^1_d(\Delta)$ for semi-algebraic splines on partitions whose edge forms have low degree (e.g. line+conic)

THANK YOU!