Commutative Algebra and Approximation Theory

Michael DiPasquale
Oklahoma State University
Colloquium

Piecewise Polynomials

Spline

A piecewise polynomial function, continuously differentiable to some order.

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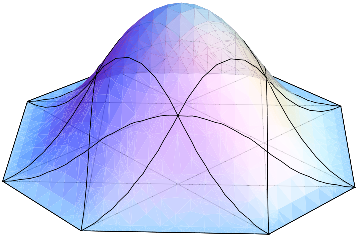

The Zwart-Powell element, a C^1 spline of degree 2

Univariate Splines

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- Subdivide $\Delta = [a, b]$ into subintervals:
  $\Delta = [a_0, a_1] \cup [a_1, a_2] \cup \cdots \cup [a_{n-1}, a_n]$
- Find a basis for the vector space $C^r_d(\Delta)$ of C^r piecewise polynomial functions on Δ with degree at most d (B-splines!)
- Find best approximation to $f(x)$ in $C^r_d(\Delta)$

Two Subintervals

$\Delta = [a_0, a_1] \cup [a_1, a_2]$ (assume WLOG $a_1 = 0$)

$$
\begin{aligned}
(f_1, f_2) \in C^r_d(\Delta) &\iff f_1^{(i)}(0) = f_2^{(i)}(0) \text{ for } 0 \le i \le r \\
&\iff x^{r+1} | (f_2 - f_1) \\
&\iff (f_2 - f_1) \in \langle x^{r+1} \rangle
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Even more explicitly:

- $f_1(x) = b_0 + b_1 x + \cdots + b_d x^d$
- $f_2(x) = c_0 + c_1 x + \cdots + c_d x^d$
- $(f_0, f_1) \in C^r_d(\Delta) \iff b_0 = c_0, \ldots, b_r = c_r$.

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- $(f_0, f_1) \in C^r_d(\Delta) \iff b_0 = c_0, \ldots, b_r = c_r$.

$$\dim C^r_d(\Delta) = \begin{cases} d+1 & \text{if } d \le r \\ (d+1) + (d-r) & \text{if } d > r \end{cases}$$

Note: $\dim C^r_d(\Delta)$ is polynomial in d for $d > r$.

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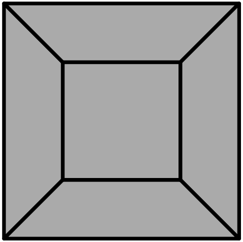

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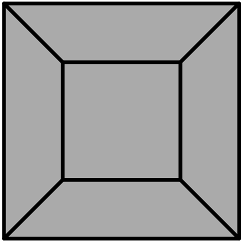


A polytopal complex

(Algebraic) Spline Criterion:

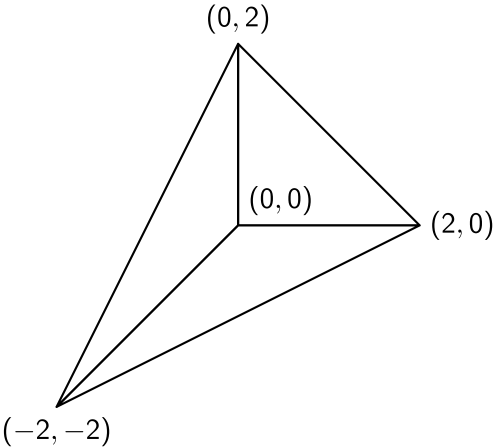
- If $\tau \in \Delta_{n-1}$, $l_\tau =$ affine form vanishing on affine span of τ
- Collection $\{f_\sigma\}_{\sigma \in \Delta_n}$ glue to $F \in C^r(\Delta) \iff$ for every pair of adjacent facets $\sigma_1, \sigma_2 \in \Delta_n$ with $\sigma_1 \cap \sigma_2 = \tau \in \Delta_{n-1}$, $l_\tau^{r+1} | (f_{\sigma_1} - f_{\sigma_2})$

Who Cares?

1. Computation of dim $C^r_d(\Delta)$ for higher dimensions initiated by [Strang '73] in connection with finite element method
2. Data fitting in approximation theory
3. [Farin '97] Computer Aided Geometric Design (CAGD) - building surfaces by splines.
4. [Payne '06] Toric Geometry - Equivariant cohomology rings of toric varieties are rings of continuous splines on the fan (under appropriate conditions).

Part I: Continuous Splines and Freeness

Continuous Splines



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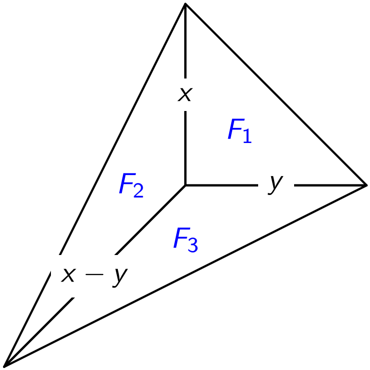

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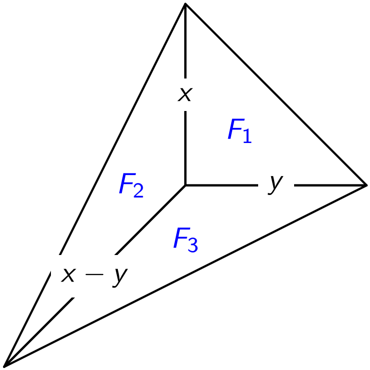
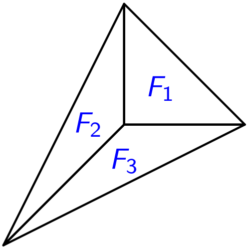


$$(F_1, F_2, F_3) \in C^0(\Delta) \iff$$
$\exists f_1, f_2, f_3$ so that

$$\begin{aligned} F_1 - F_2 &= f_1 x \\ F_2 - F_3 &= f_2(x-y) \\ F_3 - F_1 &= f_3 y \end{aligned}$$

Spline Matrix



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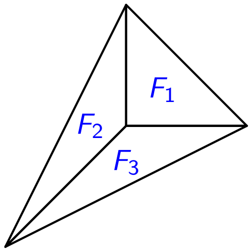

$(F_1, F_2, F_3) \in C^0(\Delta) \iff$ there are f_1, f_2, f_3 so that

$$\begin{pmatrix} 1 & -1 & 0 & x & 0 & 0 \\ 0 & 1 & -1 & 0 & x-y & 0 \\ -1 & 0 & 1 & 0 & 0 & y \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ -f_1 \\ -f_2 \\ -f_3 \end{pmatrix} = 0$$

This matrix constructed in [Billera-Rose '91].

Observations

- $C^0(\Delta)$, the kernel of this matrix, is a **graded** $\mathbb{R}[x,y]-$ module (matrix entries are homogeneous).
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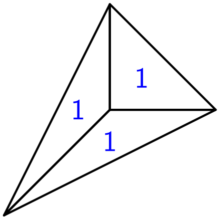
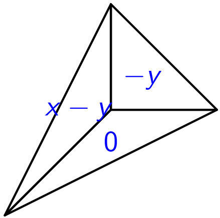
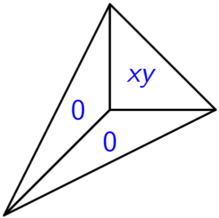
- $C^0(\Delta)_d :=$ splines of degree d
- Every spline in $C^0(\Delta)$ can be written uniquely as a polynomial combination of the three splines pictured below:







Observations, continued

$C^0(\Delta)$ is a **free** $R = \mathbb{R}[x, y]$-module generated in degrees 0,1,2.
Record degrees as $C^0(\Delta) \cong R \oplus R(-1) \oplus R(-2)$.

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$$\dim C^0(\Delta)_d = \binom{d+1}{1} + \binom{(d+1)-1}{1} + \binom{(d+1)-2}{1}$$

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$$\begin{aligned} \dim C^0(\Delta)_d = & \binom{d+1}{1} + \binom{(d+1)-1}{1} + \binom{(d+1)-2}{1} \\ = & \; 3d \text{ for } d \geq 1 \end{aligned}$$

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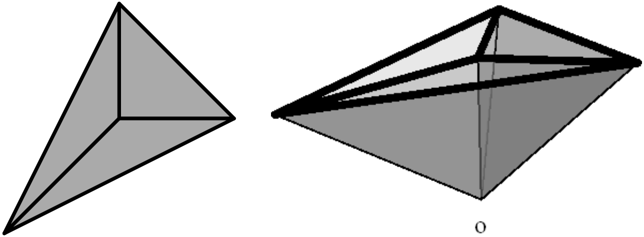
\dim C^0_d(\Delta) = & \\
\dim C^0(\widehat{\Delta})_d = \ & \binom{d+2}{2} + \binom{(d+2)-1}{2} + \binom{(d+2)-2}{2} \\
= \ & \frac{3}{2}d^2 + \frac{3}{2}d + 1 \text{ for } d \geq 0,
\end{aligned}$$

where $\widehat{\Delta}$ is the cone over Δ.

Coning Construction

- $\widehat{\Delta} \subset \mathbb{R}^{n+1}$ denotes the cone over $\Delta \subset \mathbb{R}^n$.



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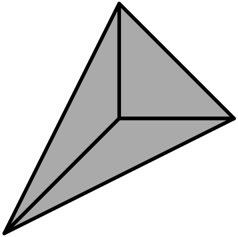

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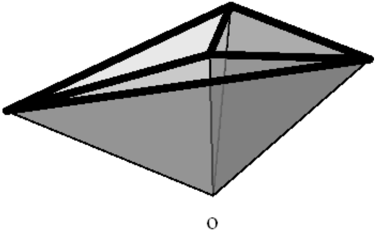

$\widehat{\Delta}$

- $C^r(\widehat{\Delta})$ is always a **graded** algebra over $S = \mathbb{R}[x_0, \ldots, x_n]$
- $C^r_d(\Delta) \cong C^r(\widehat{\Delta})_d$ [Billera-Rose '91]

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- Freeness of $C^r(\widehat{\Delta}) \implies$ straightforward computation of $\dim C^r_d(\Delta)$.

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- [Billera-Rose '92] criteria for freeness in terms of localization
- [Yuzvinsky '92] criteria for freeness in terms of sheaves on posets
- [Schenck '97] criteria for freeness in terms of homologies of a chain complex (Δ simplicial)

Face Rings of Simplicial Complexes

Face Ring of Δ

Δ a simplicial complex.

$$A_\Delta = \mathbb{R}[x_v | v \text{ a vertex of } \Delta]/I_\Delta,$$

where I_Δ is the ideal generated by monomials corresponding to non-faces.

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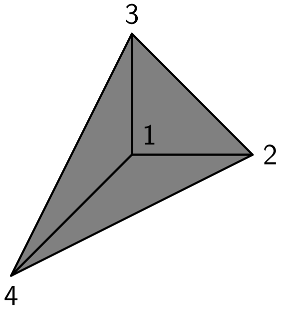

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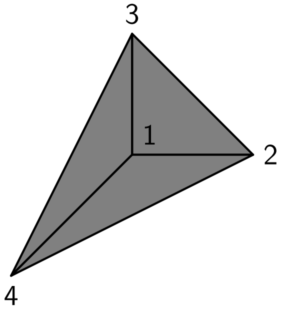

- Nonfaces are $\{1,2,3,4\}, \{2,3,4\}$
- $I_\Delta = \langle x_2 x_3 x_4 \rangle$
- $A_\Delta = \mathbb{R}[x_1, x_2, x_3, x_4]/I_\Delta$

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Moreover, if Δ is homeomorphic to a disk, then $C^0(\widehat{\Delta})$ is free.

Nonsimplicial Case

Nonfreeness for Polytopal Complexes [D. '12]

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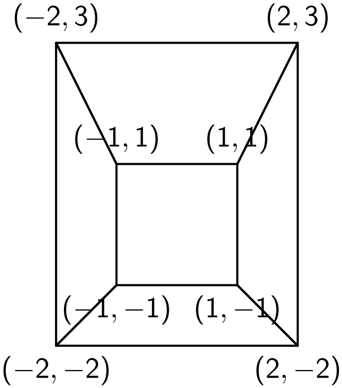

$C^0(\widehat{\Delta})$ is **not** a free module over $\mathbb{R}[x, y, z]$.

Part II: Hilbert Polynomials and Regularity

Some Graded Commutative Algebra

Given a finitely generated graded $S = \mathbb{R}[x_1, \ldots, x_n]$-module M (like $C^r(\widehat{\Delta})$).

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- If $d >> 0$, $HF(M, d) = HP(M, d)$, where $HP(M, d)$ is the **Hilbert polynomial** of M.
- Upshot: $\dim C^r_d(\Delta) = \dim C^r(\widehat{\Delta})_d$ is eventually polynomial in d (in fact, linear combination of binomial coefficients)

The Good News and the Bad News

Good news: $HP(C^r(\widehat{\Delta}), d)$ has been computed for $\Delta \subset \mathbb{R}^2$.

- Δ simplicial: [Alfeld-Schumaker '90, Hong '91], [Ibrahim-Schumaker '91]
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Bad news: $\dim C^r_d(\Delta)$ is still a mystery for small d.

- $\dim C^1_3(\Delta)$ still unknown for $\Delta \subset \mathbb{R}^2$!

Planar Hilbert Polynomials

Planar Simplicial Dimension [Alfeld-Schumaker '90]

If $\Delta \subset \mathbb{R}^2$ is a simply connected triangulation and $d \geq 3r+1$, then

$$\dim C^r_d(\Delta) = \binom{d+2}{2} + \binom{d-r+1}{2} f_1^0 - \left(\binom{d+2}{2} - \binom{r+2}{2} \right) f_0^0 + \sigma,$$

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Conjecture [Schenck '97]

Above formula holds for $d \geq 2r+1$.

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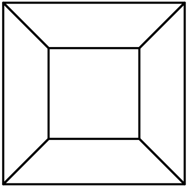
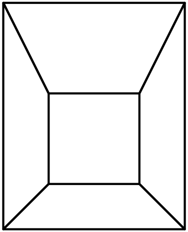


$$HP(C^0(\widehat{\Delta}), d) = \frac{5}{2}d^2 - \frac{1}{2}d + 2$$



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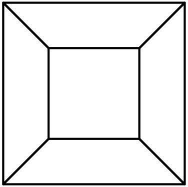

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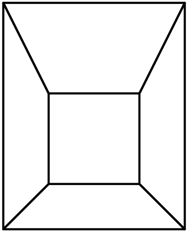

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How large does d have to be for $\dim C^r_d(\Delta) = HP(C^r(\widehat{\Delta}), d)$?

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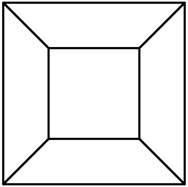
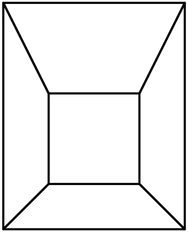
- [McDonald-Schenck '09] give formulas for coefficients of $HP(C^r(\widehat{\Delta}), d)$



$$HP(C^0(\widehat{\Delta}), d) = \frac{5}{2}d^2 - \frac{1}{2}d + 2$$



$$HP(C^0(\widehat{\Delta}), d) = \frac{5}{2}d^2 - \frac{1}{2}d + 1$$

How large does d have to be for $\dim C^r_d(\Delta) = HP(C^r(\widehat{\Delta}), d)$?
In simplicial case, $d \geq 3r + 1$ suffices.

Large degree generators

Proposition [D. '14]

Given an n-polytope $A \subset \mathbb{R}^n$ and a choice of codimension 1 face $\tau \in A_{n-1}$, there is a polytopal complex $\mathcal{P}(A)$ having A as a facet so that

1. Every codimension 1 face of A except τ is interior to $\mathcal{P}(A)$
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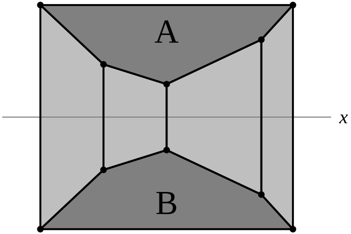

$C^r(\widehat{\Delta})$ has minimal generator of degree $4(r+1)$

A Positive Result

Agreement of Hilbert Function and Polynomial [D. '14]

$\Delta \subset \mathbb{R}^2$ a planar polytopal complex. Let $F =$ maximum number of edges of a polygon of Δ. Then

$$HP(C^r(\widehat{\Delta}), d) = \dim C^r_d(\Delta) \text{ for } d \geq (2F-1)(r+1)-1$$

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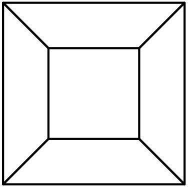

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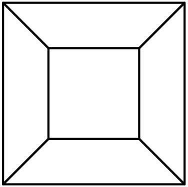

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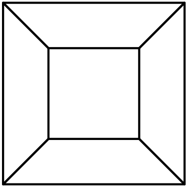

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- Macaulay2: $\dim C^0_d(\Delta) = \frac{5}{2}d^2 - \frac{1}{2}d + 2$ for $d \geq 1$

The Technique: Regularity

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Results on previous slide follow from bounding $reg(C^r(\widehat{\Delta}))$.

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- Property 1 used to break bounding $reg(LS^{r,1}(\widehat{\Delta}))$ down into a local problem by fitting into exact complexes.
- Local problem solved directly

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Main Problem:

Planar case: Lower existing regularity bounds!
Planar simplicial case: Show dim $C^r_d(\Delta) = HP(C^r(\widehat{\Delta}), d)$ for $d \geq 2r+1$.

- Regularity techniques in [D. '14] give equality in simplicial case for $d \geq 3r+2$ (one off from Alfeld-Schumaker result).
- [Schenck-Stiller '02] use vector bundle techniques on projective space to approach regularity of $C^r(\widehat{\Delta})$.

Thank You!

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