Jumping Dimensions and Projecting Polytopes

Michael DiPasquale
University of Illinois at Urbana-Champaign

Bradley University
Mathematics Colloquium

December 4, 2014

Table of Contents

Piecewise Polynomials

Spline

A piecewise polynomial function, continuously differentiable to some order.

Piecewise Polynomials

Spline

A piecewise polynomial function, continuously differentiable to some order.

Notation:

- $\mathcal{P}$: subdivision of an n-ball $\Omega \subset \mathbb{R}^n$
- $C^r(\mathcal{P})$: all splines $F : \Omega \to \mathbb{R}$ continuously differentiable of order r
- **Degree** of a spline: max degree of polynomials it restricts to
- $C^r_d(\mathcal{P})$: splines of degree $\leq d$ on $\mathcal{P}$

Application: Approximation

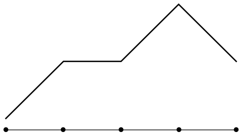
Splines are a cornerstone of **approximation theory** - used to approximate complicated functions.

Application: Approximation

Splines are a cornerstone of **approximation theory** - used to approximate complicated functions.
Low degree splines are used in Calc 1 to approximate integrals.

Application: Approximation

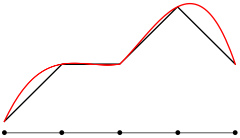
Splines are a cornerstone of **approximation theory** - used to approximate complicated functions.
Low degree splines are used in Calc 1 to approximate integrals.



Graph of piecewise linear function

Application: Approximation

Splines are a cornerstone of **approximation theory** - used to approximate complicated functions.
Low degree splines are used in Calc 1 to approximate integrals.

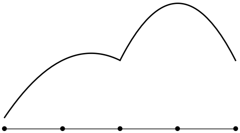


Trapezoid Rule

Application: Approximation

Splines are a cornerstone of **approximation theory** - used to approximate complicated functions.

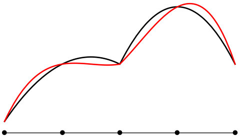
Low degree splines are used in Calc 1 to approximate integrals.



Graph of piecewise quadratic function

Application: Approximation

Splines are a cornerstone of **approximation theory** - used to approximate complicated functions.
Low degree splines are used in Calc 1 to approximate integrals.



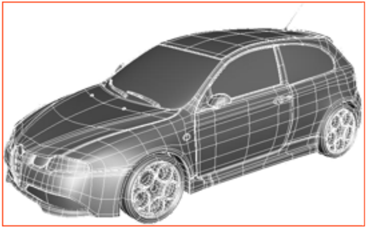
Simpson's Rule

Application: Computer-Aided Design

Term **spline** originated in shipbuilding - referred to flexible wooden strips anchored at several points.
Today, splines are used extensively to create models by interpolating datapoints.

Application: Computer-Aided Design

Term **spline** originated in shipbuilding - referred to flexible wooden strips anchored at several points.
Today, splines are used extensively to create models by interpolating datapoints.



Calculus Exercise: I

For what value of c is the following function continuous?

$$f(x) = \begin{cases} x^2 + x + c & -1 \leq x < 0 \\ 2x + 1 & 0 \leq x \leq 1 \end{cases}$$

Calculus Exercise: I

For what value of c is the following function continuous?

$$f(x) = \begin{cases} x^2 + x + c & -1 \leq x < 0 \\ 2x + 1 & 0 \leq x \leq 1 \end{cases}$$

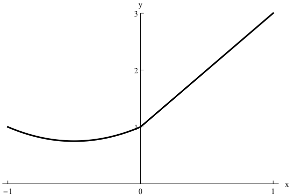
- Answer: $c = 1$
- With $c = 1$, $f(x)$ is a C^0 **spline** on the subdivision $I = [-1, 0] \cup [0, 1]$ of $[-1, 1]$.
- Notation: $f \in C^0_2(I)$

Calculus Exercise: I

For what value of c is the following function continuous?

$$f(x) = \begin{cases} x^2 + x + c & -1 \leq x < 0 \\ 2x + 1 & 0 \leq x \leq 1 \end{cases}$$

- Answer: $c = 1$
- With $c = 1$, $f(x)$ is a C^0 **spline** on the subdivision $I = [-1, 0] \cup [0, 1]$ of $[-1, 1]$.
- Notation: $f \in C^0_2(I)$



Graph of f

Calculus Exercise II

For what value of b is the following function differentiable?

$$g(x) = \begin{cases} x^2 + bx + 1 & -1 \leq x < 0 \\ 2x + 1 & 0 \leq x \leq 1 \end{cases}$$

Calculus Exercise II

For what value of b is the following function differentiable?

$$g(x) = \begin{cases} x^2 + bx + 1 & -1 \leq x < 0 \\ 2x + 1 & 0 \leq x \leq 1 \end{cases}$$

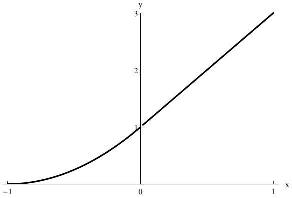
- Answer: $b = 2$
- With $b = 2$, $g(x)$ is a C^1 **spline** on $I = [-1, 0] \cup [0, 1]$.
- Notation: $g \in C^1_2(I)$

Calculus Exercise II

For what value of b is the following function differentiable?

$$g(x) = \begin{cases} x^2 + bx + 1 & -1 \leq x < 0 \\ 2x + 1 & 0 \leq x \leq 1 \end{cases}$$

- Answer: $b = 2$
- With $b = 2$, $g(x)$ is a C^1 **spline** on $I = [-1, 0] \cup [0, 1]$.
- Notation: $g \in C^1_2(I)$



Graph of g

Counting Univariate Splines

$I = [-1, 0] \cup [0, 1]$

$$h(x) = \begin{cases} ax + b & -1 \leq x < 0 \\ cx + d & 0 \leq x \leq 1 \end{cases}$$

Which of the coefficients a, b, c, d can be chosen freely if $h(x)$ is required to be continuous?

Counting Univariate Splines

$I = [-1, 0] \cup [0, 1]$

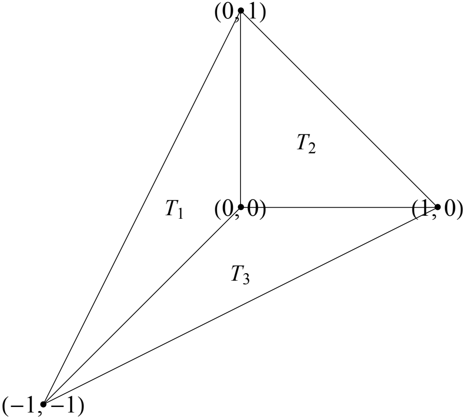
$$h(x) = \begin{cases} ax + b & -1 \leq x < 0 \\ cx + d & 0 \leq x \leq 1 \end{cases}$$

Which of the coefficients a, b, c, d can be chosen freely if $h(x)$ is required to be continuous?

- Must have $b = d$
- So free to determine a, b, c
- $C_1^0(I)$ is a **three dimensional** vector space

Counting Bivariate Splines

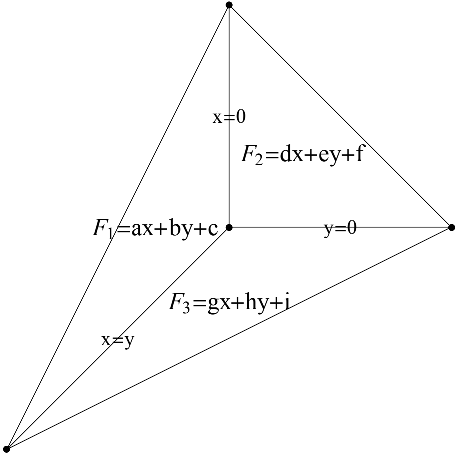
Δ = union of three triangles below



Δ

Counting Bivariate Splines

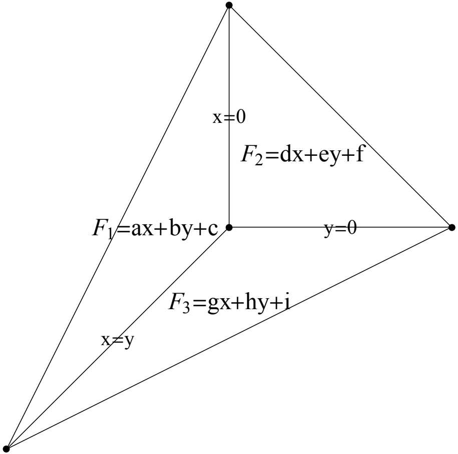
Δ = union of three triangles below



Candidate for $F \in C^0_1(\Delta)$

Counting Bivariate Splines

Δ = union of three triangles below



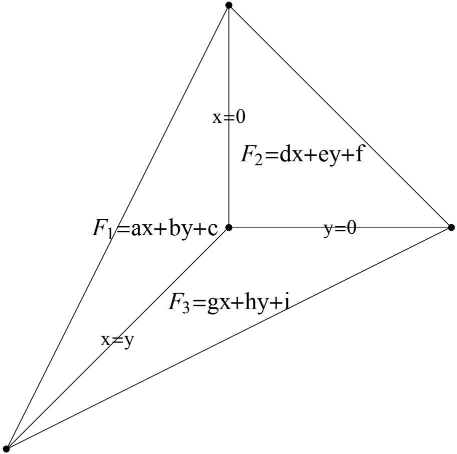
Continuity $\Longrightarrow$

$$\begin{aligned} b &= e \\ c = f &= i \\ d &= g \\ a + b &= g + h \end{aligned}$$

Candidate for $F \in C^0_1(\Delta)$

Counting Bivariate Splines

Δ = union of three triangles below



Candidate for $F \in C_1^0(\Delta)$

Continuity $\implies$

$$b = e$$
$$c = f = i$$
$$d = g$$
$$a + b = g + h$$

a, b, c, d determine e, f, g, h, i $\implies$ $C_1^0(\Delta)$ is 4-dim vector space

Interlude: Vector Spaces

A **vector space** V over the real numbers looks like $\mathbb{R}^n$
You can add vectors and multiply them by scalars.

Example: $\mathbb{R}^2$

- **Add** vectors: $(a, b) + (c, d) = (a + c, b + d)$
- **Multiply** vectors by scalars: $r(a, b) = (ra, rb)$, where r is a real number.

A **linear combination** of vectors $v_1, \ldots, v_k$ is a sum

$$r_1 v_1 + \cdots + r_k v_k,$$

where $r_1, \ldots, r_k$ are real numbers.

Basis and Dimension

Basis: $v_1, \ldots, v_k \in V$ is a basis if any vector can be written uniquely as a linear combination of $v_1, \ldots, v_k$.

Example:

- Standard basis of $\mathbb{R}^2$: $\{(1,0),(0,1)\}$
- Different basis of $\mathbb{R}^2$: $\{(1,-1),(1,1)\}$
- Not a basis of $\mathbb{R}^2$: $\{(1,0),(0,1),(1,1)\}$

Dimension of the vector space V is the number of vectors in a basis. For example:

- $\dim \mathbb{R}^2 = 2$
- $\dim \mathbb{R}^n = n$

Notation: $\dim V$ means dimension of V.

Vector Spaces of Splines

Vector Spaces of Splines

For any subdivision $\mathcal{P}$ and any choice of r and d, $C^r_d(\mathcal{P})$ is a vector space.

Reason: Adding splines and multiplying them by scalars does not effect their degree or existence of derivatives.

Main Question

Given $\mathcal{P}$ a subdivision of ball in $\mathbb{R}^n$.

Main Questions

Q1 What is $\dim C^r_d(\mathcal{P})$ in terms of r and the data of the subdivision?

Q2 Can we find a basis for $C^r_d(\mathcal{P})$?

Main Question

Given $\mathcal{P}$ a subdivision of ball in $\mathbb{R}^n$.

Main Questions

- Q1 What is dim $C^r_d(\mathcal{P})$ in terms of r and the data of the subdivision?
- Q2 Can we find a basis for $C^r_d(\mathcal{P})$?

Known results:

- If I is a subdivision of an interval in $\mathbb{R}$ then Q1 and Q2 are standard results
- If Δ is a triangulation in $\mathbb{R}^2$ and $d \geq 3r+2$, Q1 and Q2 are known [Alfeld-Schumaker '90]
- If $\mathcal{P}$ is a polygonal subdivision in $\mathbb{R}^2$, Q1 is known for large d [McDonald-Schenck '09]

Main Question

Given $\mathcal{P}$ a subdivision of ball in $\mathbb{R}^n$.

Main Questions

Q1 What is $\dim C^r_d(\mathcal{P})$ in terms of r and the data of the subdivision?

Q2 Can we find a basis for $C^r_d(\mathcal{P})$?

Known results:

- If I is a subdivision of an interval in $\mathbb{R}$ then Q1 and Q2 are standard results
- If Δ is a triangulation in $\mathbb{R}^2$ and $d \geq 3r+2$, Q1 and Q2 are known [Alfeld-Schumaker '90]
- If $\mathcal{P}$ is a polygonal subdivision in $\mathbb{R}^2$, Q1 is known for large d [McDonald-Schenck '09]

If Δ is a triangulation in $\mathbb{R}^2$, $\dim C^1_3(\Delta)$ is not known in general.

When $r = 0, d = 1$ (piecewise linear) we'll see:

- Nice answers for Q1 and Q2 if $\mathcal{P}$ is subdivision I of an interval in $\mathbb{R}^1$
- Nice answer for Q1 and Q2 if $\mathcal{P}$ is a triangulation Δ in $\mathbb{R}^2$
- No simple answer for Q1 or Q2 if $\mathcal{P}$ is a polygonal subdivision in $\mathbb{R}^2$ (dimensions may jump for certain configurations)

Univariate Piecewise Linear Functions

Theorem

If I is a subdivision of an interval with v vertices, then

1. $\dim C^0_1(I) = v$
2. *A basis for $C^0_1(I)$ is given by 'tent' functions*

Univariate Piecewise Linear Functions

Theorem

If I is a subdivision of an interval with v vertices, then

1. $\dim C_1^0(I) = v$
2. *A basis for $C_1^0(I)$ is given by 'tent' functions*

Proof of part 1: PL function determined uniquely by value on **vertices**

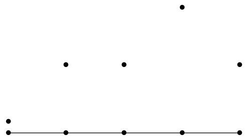
Univariate Piecewise Linear Functions

Theorem

If I is a subdivision of an interval with v vertices, then

1. $\dim C^0_1(I) = v$
2. *A basis for $C^0_1(I)$ is given by 'tent' functions*

Proof of part 1: PL function determined uniquely by value on **vertices**



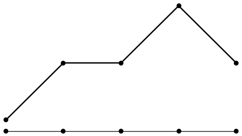
Univariate Piecewise Linear Functions

Theorem

If I is a subdivision of an interval with v vertices, then

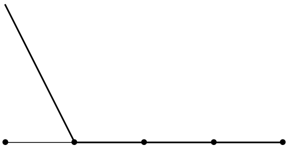
1. $\dim C^0_1(I) = v$
2. *A basis for $C^0_1(I)$ is given by 'tent' functions*

Proof of part 1: PL function determined uniquely by value on **vertices**



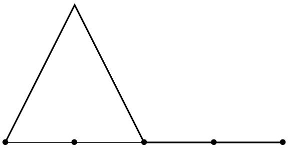
Tent Functions 1

'Courant functions' or 'tent functions' are 1 at a chosen vertex and 0 at all others:



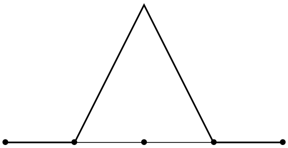
Tent Functions 1

'Courant functions' or 'tent functions' are 1 at a chosen vertex and 0 at all others:



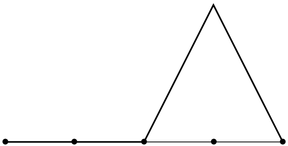
Tent Functions 1

'Courant functions' or 'tent functions' are 1 at a chosen vertex and 0 at all others:



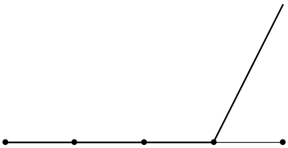
Tent Functions 1

'Courant functions' or 'tent functions' are 1 at a chosen vertex and 0 at all others:



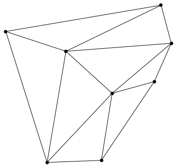
Tent Functions 1

‘Courant functions’ or ‘tent functions’ are 1 at a chosen vertex and 0 at all others:



Onward to 2 dimensions

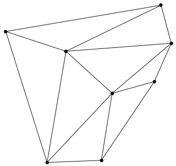
- Have many more choices for a subdivision of a 2-ball
- A natural choice: Triangulations!
- Important: Only allow triangles to meet along full edges
- Δ = triangulation of a 2-ball, with v vertices, e edges, f faces (triangles)



A triangulation Δ with $v = 8, e = 15, f = 8$

Onward to 2 dimensions

- Have many more choices for a subdivision of a 2-ball
- A natural choice: Triangulations!
- Important: Only allow triangles to meet along full edges
- $\Delta =$ triangulation of a 2-ball, with v vertices, e edges, f faces (triangles)



A triangulation Δ with $v = 8, e = 15, f = 8$

What kinds of PL functions are there on Δ?

Theorem

If $\Delta \subset \mathbb{R}^2$ is a subdivision of a disk with v vertices, then

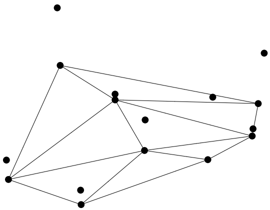
1. $\dim C^0_1(\Delta) = v$
2. *A basis for $C^0_1(\Delta)$ is given by 'tent' functions*

Theorem

If $\Delta \subset \mathbb{R}^2$ is a subdivision of a disk with v vertices, then

1. $\dim C^0_1(\Delta) = v$
2. *A basis for $C^0_1(\Delta)$ is given by 'tent' functions*

Proof of part 1: PL function on Δ uniquely determined by value at vertices.

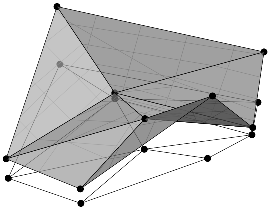


Theorem

If $\Delta \subset \mathbb{R}^2$ is a subdivision of a disk with v vertices, then

1. $\dim C^0_1(\Delta) = v$
2. *A basis for $C^0_1(\Delta)$ is given by 'tent' functions*

Proof of part 1: PL function on Δ uniquely determined by value at vertices.

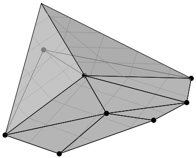


Tent Functions 2

Just as before, Courant functions are 1 at a chosen vertex and 0 on other vertices.

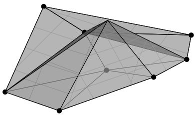
Tent Functions 2

Just as before, Courant functions are 1 at a chosen vertex and 0 on other vertices.



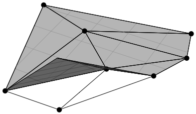
Tent Functions 2

Just as before, Courant functions are 1 at a chosen vertex and 0 on other vertices.



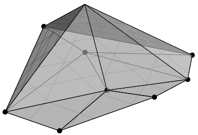
Tent Functions 2

Just as before, Courant functions are 1 at a chosen vertex and 0 on other vertices.



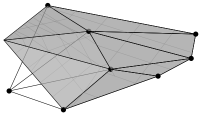
Tent Functions 2

Just as before, Courant functions are 1 at a chosen vertex and 0 on other vertices.



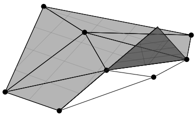
Tent Functions 2

Just as before, Courant functions are 1 at a chosen vertex and 0 on other vertices.



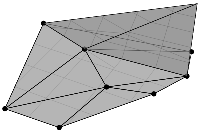
Tent Functions 2

Just as before, Courant functions are 1 at a chosen vertex and 0 on other vertices.



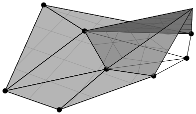
Tent Functions 2

Just as before, Courant functions are 1 at a chosen vertex and 0 on other vertices.



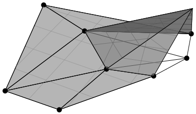
Tent Functions 2

Just as before, Courant functions are 1 at a chosen vertex and 0 on other vertices.



Tent Functions 2

Just as before, Courant functions are 1 at a chosen vertex and 0 on other vertices.

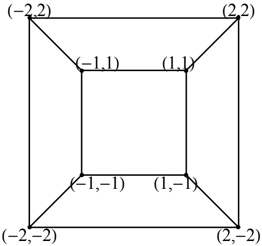


- Note: $\dim C^0_1(I)$ and $\dim C^0_1(\Delta)$ only depended on number of vertices.
- No dependence on geometry!

Polygons

What if we use a subdivision $\mathcal{P}$ consisting of convex polygons instead of triangles?

- Convex: line segment joining any two points of the polygon is also inside the polygon.
- Call this a **polygonal subdivision**
- f, e, v stay the same



A polygonal subdivision $\mathcal{P}$ with $f = 5, e = 12, v = 8$

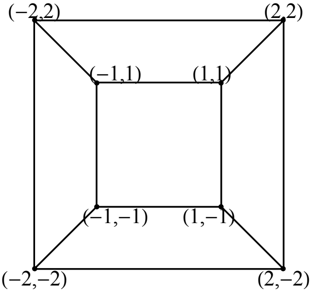
Does $\dim C^0_1(\mathcal{P}) = v$?

Theorem

If $\mathcal{P} \subset \mathbb{R}^2$ is a polygonal subdivision of a disk, $\dim C^0_1(\mathcal{P})$ depends on geometry of $\mathcal{P}$.

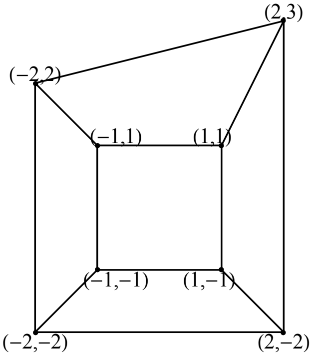
- $\dim C^0_1(\mathcal{P}) < v$ unless $\mathcal{P}$ is a triangulation
- Lose tent functions!

Proof by Example



$\mathcal{Q}_1$

$\dim C^0_1(\mathcal{Q}_1) = 4$

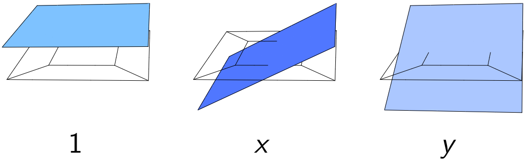


$\mathcal{Q}_2$

$\dim C^0_1(\mathcal{Q}_2) = 3$

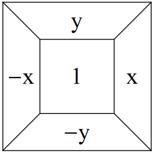
Trivial PL Functions

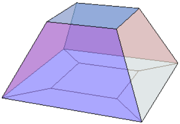
- A **trivial** PL function on $\mathcal{P}$ has the same linear function on each face.
- dim(trivial splines on $\mathcal{P}$) $= 3$ **always**, with basis $1, x, y$.



NonTrivial PL Functions

- **Nontrivial** PL function on $\mathcal{P}$ has at least two different polynomials on different faces.
- One **nontrivial** PL function on $\mathcal{Q}_1$, whose graph is below:





When you move to $\mathcal{Q}_2$ you lose this function!

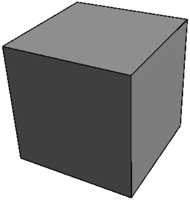
Dependence on Geometry

More explicit: Polygonal subdivisions coming from polytopes have special PL functions.

Dependence on Geometry

More explicit: Polygonal subdivisions coming from polytopes have special PL functions.

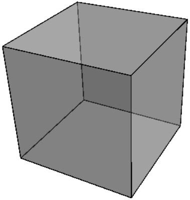
Here's a cube



Dependence on Geometry

More explicit: Polygonal subdivisions coming from polytopes have special PL functions.

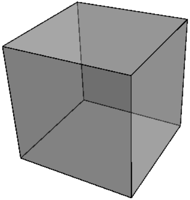
Make it transparent



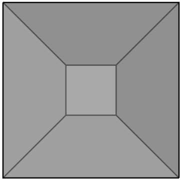
Dependence on Geometry

More explicit: Polygonal subdivisions coming from polytopes have special PL functions.

Make it transparent



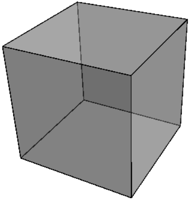
Now look in one of the faces:



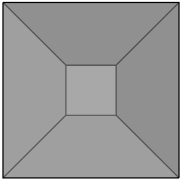
Dependence on Geometry

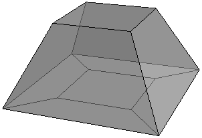
More explicit: Polygonal subdivisions coming from polytopes have special PL functions.

Make it transparent



Now look in one of the faces:

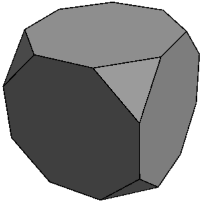




The nontrivial PL function is a 'deformed cube'

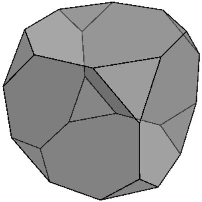
More Interesting Example

Chop off cube corners



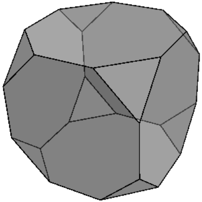
More Interesting Example

Make it transparent

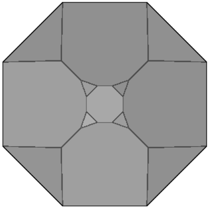


More Interesting Example

Make it transparent

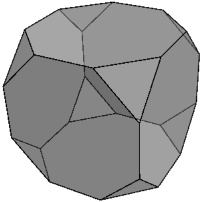


Look into an octagonal face:

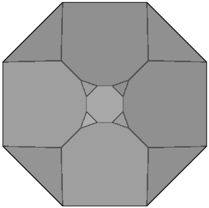


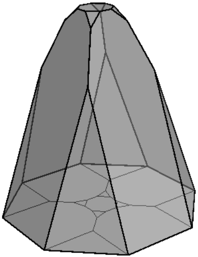
More Interesting Example

Make it transparent



Look into an octagonal face:





Nontrivial PL function is 'deformed' version of truncated cube

Where to now?

- We've seen that $\dim C^0_1(\mathcal{P})$ is subtle for polygonal subdivisions.
- What about $\dim C^r_d(\mathcal{P})$, where $r > 0$, $d > 1$?

Where to now?

- We've seen that $\dim C^0_1(\mathcal{P})$ is subtle for polygonal subdivisions.
- What about $\dim C^r_d(\mathcal{P})$, where $r > 0$, $d > 1$?
- For fixed $\mathcal{P}$ and d large, $\dim C^r_d(\mathcal{P})$ is a polynomial in d!
- For small d, $\dim C^r_d(\mathcal{P})$ may not agree with this polynomial.

Some Dimension Formulas

$I \subset \mathbb{R}$ subdivision with v vertices, e edges, v^0 interior vertices.

$$\dim_{\mathbb{R}} C^r_d(I) = \begin{cases} d+1 & d \leq r \\ e(d+1) - v^0(r+1) & d > r \end{cases}$$

Some Dimension Formulas

$I \subset \mathbb{R}$ subdivision with v vertices, e edges, v^0 interior vertices.

$$\dim_{\mathbb{R}} C^r_d(I) = \begin{cases} d+1 & d \leq r \\ e(d+1) - v^0(r+1) & d > r \end{cases}$$

$\Delta \subset \mathbb{R}^2$ triangulation: f triangles, e^0 interior edges, v^0 interior vertices.

$$\dim C^0_d(\Delta) = f\frac{(d+2)(d+1)}{2} - e^0(d+1) + v^0$$

for all $d \geq 0$.

Conclusion

- Computations of dim $C^r_d(\mathcal{P})$ for $r = 0$, $d = 1$ can be difficult for polygonal subdivisions
- Dimension depends on combinatorial and geometric data of subdivision

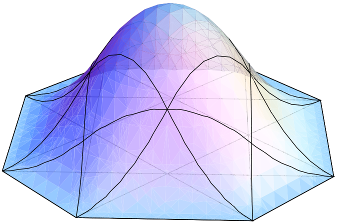
Conclusion

- Computations of dim $C^r_d(\mathcal{P})$ for $r = 0$, $d = 1$ can be difficult for polygonal subdivisions
- Dimension depends on combinatorial and geometric data of subdivision

Two main approaches to compute dim $C^r_d(\mathcal{P})$ in general.

- Analytic approach - deals explicitly with coefficients of splines over triangulations
- Used by Alfeld-Schumaker [Alfeld-Schumaker '90], others
- Algebraic approach pioneered in [Billera '88] - uses tools of homological and commutative algebra

THANK YOU!



References I

P. Alfeld, L. Schumaker, *On the dimension of bivariate spline spaces of smoothness r and degree $d = 3r + 1$*, Numer. Math. **57** (1990) 651-661.

L. Billera, *Homology of Smooth Splines: Generic Triangulations and a Conjecture of Strang*, Trans. Amer. Math. Soc. **310**, 325-340 (1988).

L. Billera, L. Rose, *A Dimension Series for Multivariate Splines*, Discrete Comput. Geom. **6**, 107-128 (1991).

G. Farin, *Curves and Surfaces for Computer Aided Geometric Design*, 4th ed., Academic Press, Boston, 1997.

D. Hong, Spaces of bivariate spline functions over triangulation, Approx. Theory Appl. 7 (1991), 56-75.

T. McDonald, H. Schenck, *Piecewise Polynomials on Polyhedral Complexes*, Adv. in Appl. Math. **42** , no. 1, 82-93 (2009).