Counting Piecewise Linear Functions

Michael DiPasquale

Two Calculus Exercises

Univariate PL Functions

Bivariate PL Functions

Static Equilibrium

Where to now?

Counting Piecewise Linear Functions

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Piecewise Polynomials

Spline

A piecewise polynomial function, continuously differentiable to some order.

Some Context: Splines in Calculus 1

Low degree splines are used in Calc 1 to approximate integrals.

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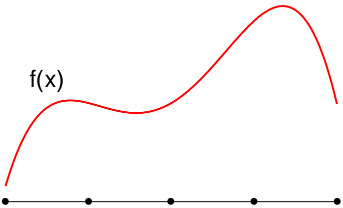

Graph of a function

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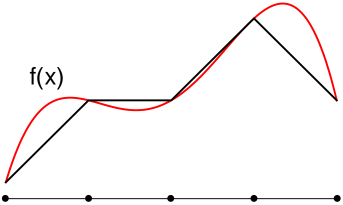

Trapezoid Rule

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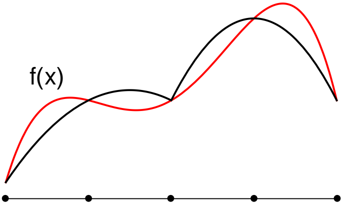
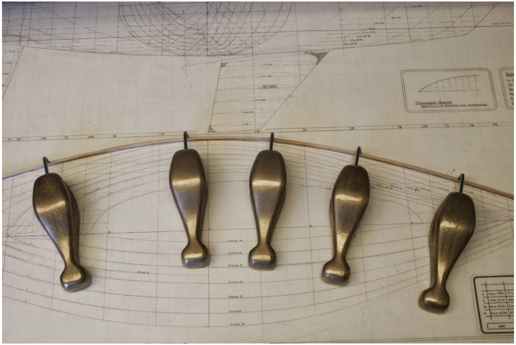


Simpson's Rule

Origin: Ship-Building

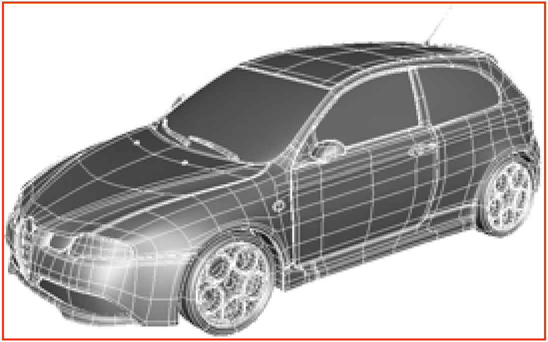
Term **spline** originated in shipbuilding - referred to flexible wooden strips anchored at several points.



Source: *http://technologycultureboats.blogspot.com/2014/12/gustave-caillebotte-and-curves.html*

Application: Computer-Aided Geometric Design

Today, splines are used extensively to create models by interpolating datapoints (CAGD).



Source: *http://www.tsplines.com/products/what-are-t-splines.html*

Calculus Exercise: I

For what value of c is the following function continuous?

$$f(x) = \begin{cases} x^2 + x + c & -1 \leq x < 0 \\ 2x + 1 & 0 \leq x \leq 1 \end{cases}$$

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- Answer: $c = 1$
- With $c = 1$, $f(x)$ is a C^0 **spline** on the subdivision $I = [-1, 0] \cup [0, 1]$ of $[-1, 1]$.
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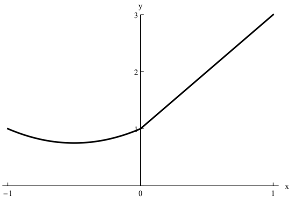

Graph of f

Calculus Exercise II

For what value of b is the following function differentiable?

$$g(x) = \begin{cases} x^2 + bx + 1 & -1 \leq x < 0 \\ 2x + 1 & 0 \leq x \leq 1 \end{cases}$$

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$$g(x) = \begin{cases} x^2 + bx + 1 & -1 \leq x < 0 \\ 2x + 1 & 0 \leq x \leq 1 \end{cases}$$

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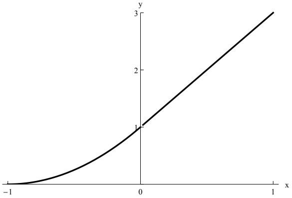

Graph of g

Counting Univariate PL Functions

$I = [-1, 0] \cup [0, 1]$

$$h(x) = \begin{cases} ax + b & -1 \leq x < 0 \\ cx + d & 0 \leq x \leq 1 \end{cases}$$

Which of the coefficients a, b, c, d can be chosen freely if $h(x)$ is required to be continuous?

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Which of the coefficients a, b, c, d can be chosen freely if $h(x)$ is required to be continuous?

- Must have $b = d$
- So free to determine a, b, c
- $C^0_1(I)$ is a **three dimensional** vector space

Dimension Question

Suppose I is a subdivision of an interval into a union of subintervals.

- What is $\dim C^0_1(I)$?
- Can we find a basis for $C^0_1(I)$?

Counting Univariate PL Functions

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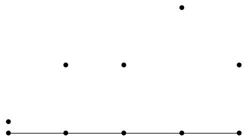

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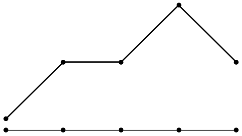

Tent Functions

A basis for $C^0_1(I)$ is given by 'Courant functions' or 'tent functions' are 1 at a chosen vertex and 0 at all others.

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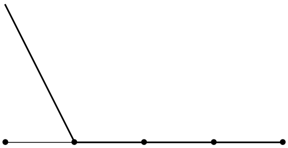

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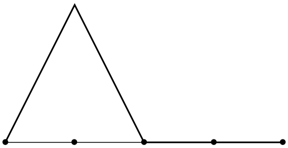

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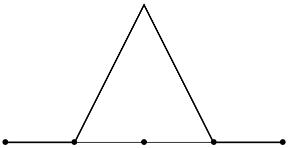

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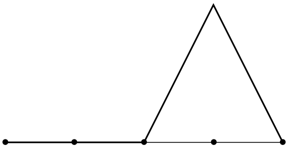

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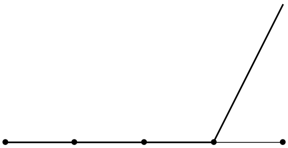
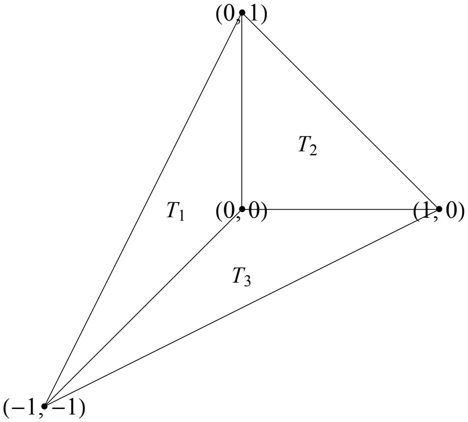


Counting Bivariate PL Functions

Δ = union of three triangles below



Δ

Counting Bivariate PL Functions

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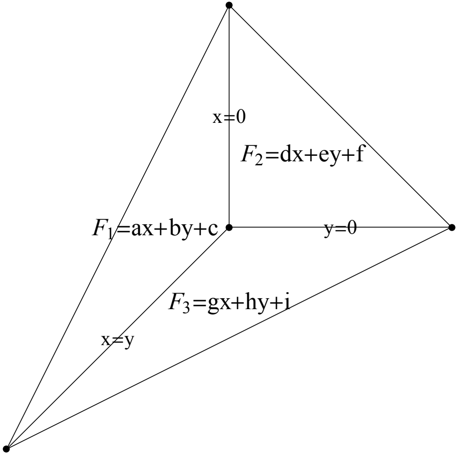

Candidate for $F \in C^0_1(\Delta)$

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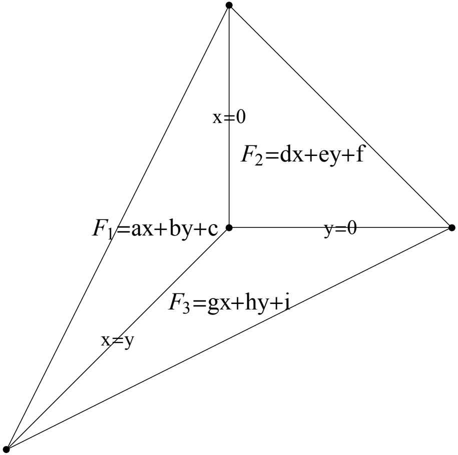

Continuity $\Longrightarrow$

$$\begin{aligned} b &= e \\ c = f &= i \\ d &= g \\ a + b &= g + h \end{aligned}$$

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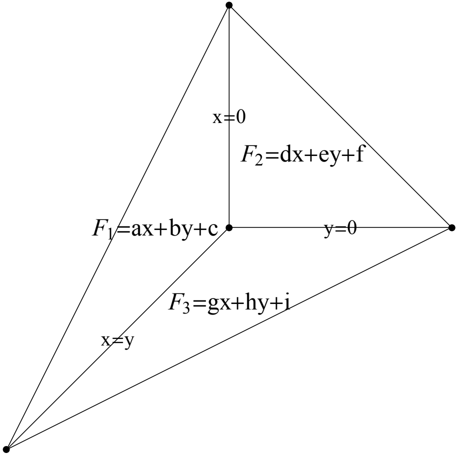

Candidate for $F \in C^0_1(\Delta)$

Continuity $\implies$

$$b = e$$
$$c = f = i$$
$$d = g$$
$$a + b = g + h$$

a, b, c, d determine e, f, g, h, i $\implies$ $C^0_1(\Delta)$ is 4-dim vector space

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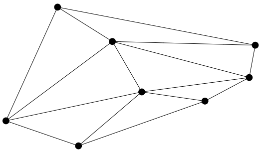

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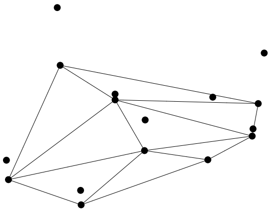

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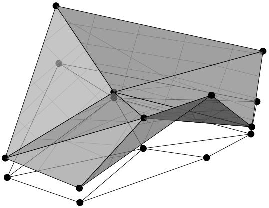

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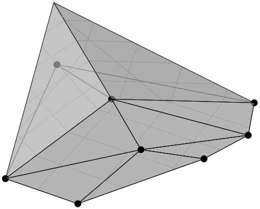

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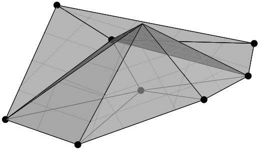

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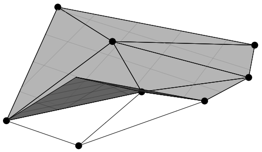

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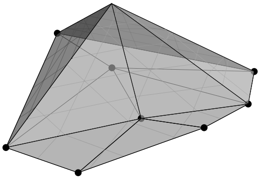

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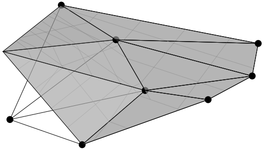

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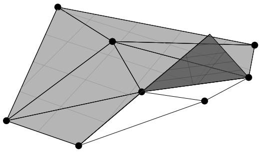

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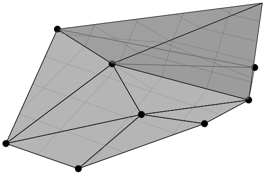

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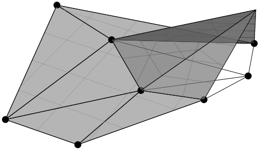

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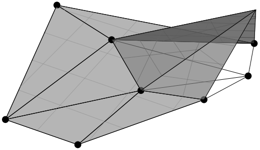

- Note: $\dim C^0_1(I)$ and $\dim C^0_1(\Delta)$ only depended on number of vertices.
- No dependence on geometry!

Polygonal Subdivisions

What if we use a polygonal subdivision instead of a triangulation?

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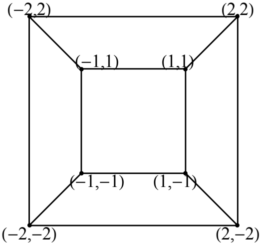

A polygonal subdivision $\mathcal{P}$

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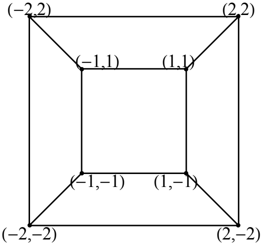


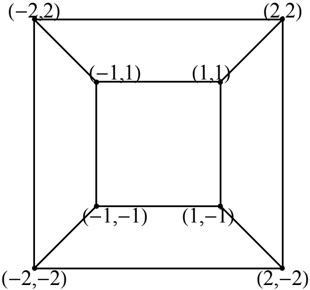
A polygonal subdivision $\mathcal{P}$

Does $\dim C_1^0(\mathcal{P}) = v$?

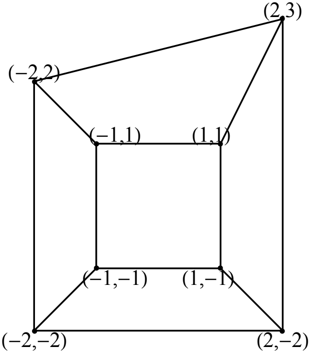
If $\mathcal{P} \subset \mathbb{R}^2$ is a polygonal subdivision, dim $C^0_1(\mathcal{P})$ depends on geometry of $\mathcal{P}$!

- dim $C^0_1(\mathcal{P}) < v$ unless $\mathcal{P}$ is a triangulation
- Lose tent functions!

Proof by Example



$\mathcal{Q}_1$



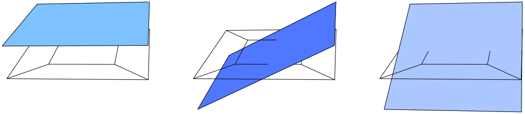
$\mathcal{Q}_2$

$\dim C^0_1(\mathcal{Q}_1) = 4$

$\dim C^0_1(\mathcal{Q}_2) = 3$

Trivial PL Functions

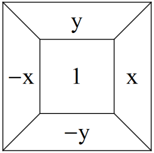
- A **trivial** PL function on $\mathcal{P}$ has the same linear function on each face.
- $\dim(\text{trivial splines on } \mathcal{P}) = 3$ **always**, with basis $1, x, y$.

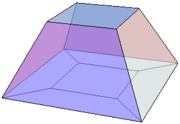


1 $\quad$ x $\quad$ y

NonTrivial PL Functions

- **Nontrivial** PL function on has at least two different linear functions on different faces.
- One **nontrivial** PL function on $\mathcal{Q}_1$, whose graph is below:





When you move to $\mathcal{Q}_2$ you lose this function!

Dependence on Geometry

More explicit: Polygonal subdivisions coming from projections of polytopes have special PL functions.

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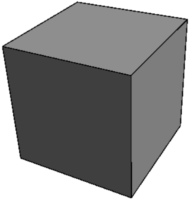
Here's a cube



Dependence on Geometry

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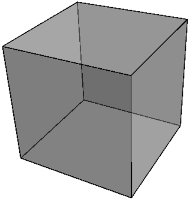
Make it transparent



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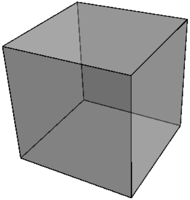
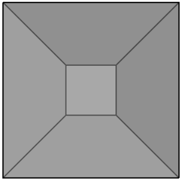


Now look in one of the faces:



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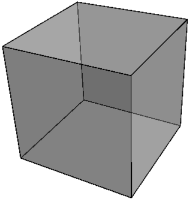

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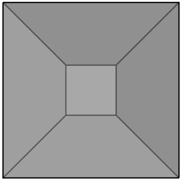
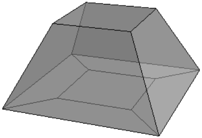
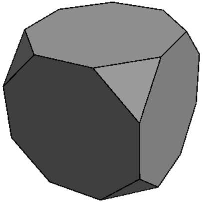




The nontrivial PL function is a 'deformed cube'

More Interesting Example

Chop off cube corners



More Interesting Example

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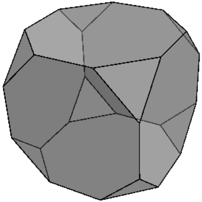

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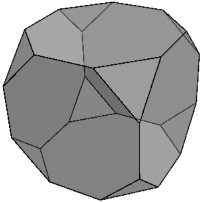

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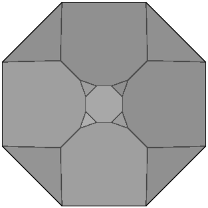

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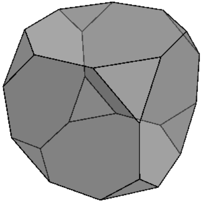

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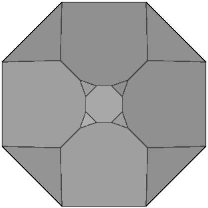

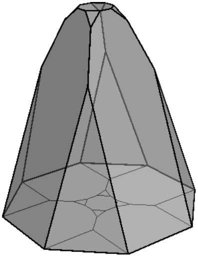

Nontrivial PL function is 'deformed' version of truncated cube

Tension and Compression on Polygonal Subdivisions

- Planar framework of bars and joints given by edges and vertices of polygonal subdivision

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Counting Piecewise Linear Functions

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Two Calculus Exercises

Univariate PL Functions

Bivariate PL Functions

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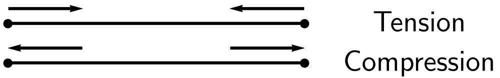

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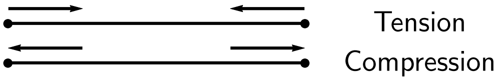

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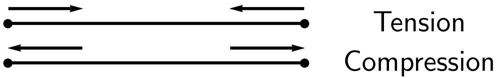

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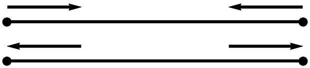

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- Force $\omega_{ij}(p_j - p_i)$ at p_i
- Force $\omega_{ij}(p_i - p_j)$ at p_j
- $\omega_{ij} < 0 \implies$ tension
- $\omega_{ij} > 0 \implies$ compression

Self-Stress

A **self-stress** on a framework is an assignment of scalars ω_{ij} along the edges e_{ij} satisfying

$$\sum_{p_j \text{ adjacent to } p_i} \omega_{ij}(p_j - p_i) = 0.$$

for every interior vertex p_i.

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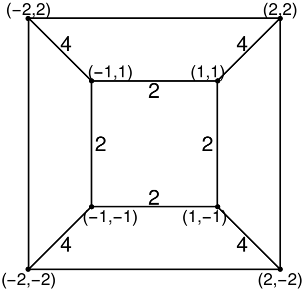

A nontrivial self-stress on $\mathcal{P}_1$

Matrix for Self-Stresses

Self-stresses are the null space of a matrix.

$$\begin{array}{c} \\ p_1 \\ \\ p_2 \\ \\ p_3 \\ \\ p_4 \\ \\ \end{array}
\begin{array}{c} \begin{array}{cccccccc} 12 & 23 & 34 & 14 & 15 & 26 & 37 & 48 \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & -1 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \end{array}$$

Maxwell's Observation [Crapo-Whiteley '93]

Nontrivial stresses are in 1-1 correspondence (almost) with nontrivial PL functions on $\mathcal{P}$!

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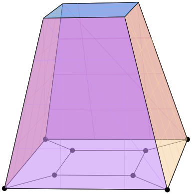

Start with graph

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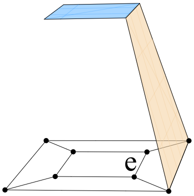

Restrict to faces adjacent to a single edge e

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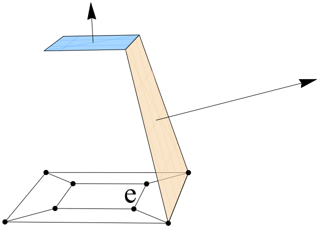

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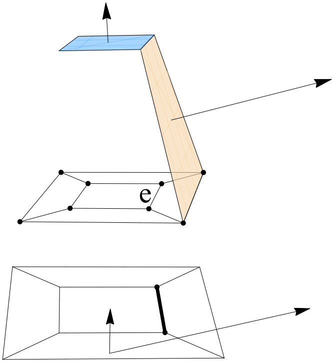

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Take normals (z-component$= 1$)

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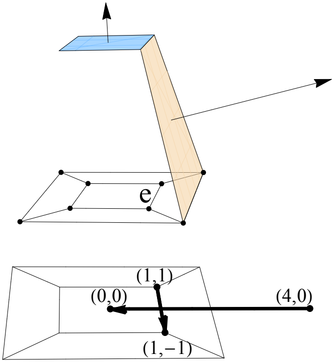

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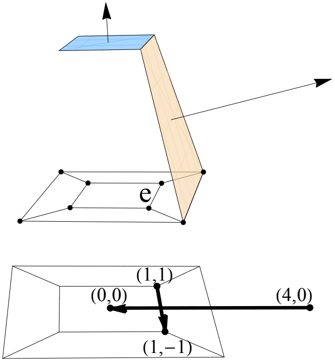

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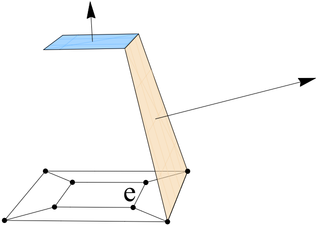

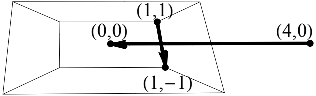

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Sign of ω_e **depends on orientation**.

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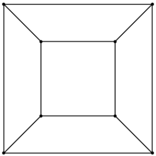
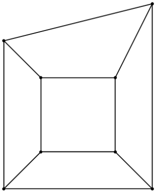


$\mathcal{P}_1$ **is not** independent



$\mathcal{P}_2$ **is** independent

Summary so far

We've seen:

- $\dim C^0_1(I) = v$ for a subdivision I of an interval
- $\dim C^0_1(\Delta) = v$ for a planar triangulation Δ
- $\dim C^0_1(\mathcal{P})$ for a planar polygonal subdivision $\mathcal{P}$ relies on counting the number of ways polygonal surfaces can project onto $\mathcal{P}$
- Equivalently, $\dim C^0_1(\mathcal{P})$ relies on computing the dimension of the vector space of self-stresses on $\mathcal{P}$.

Where to now?

What about $\dim C^r_d(\mathcal{P})$, where $r > 0$, $d > 1$?

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- For fixed $\mathcal{P}$ and d large, $\dim C^r_d(\mathcal{P})$ is a polynomial in d!
- For small d, $\dim C^r_d(\mathcal{P})$ may not agree with this polynomial.

Univariate Dimension Formula

Suppose I is a subdivision of an interval with v^0 interior vertices and e edges. Then

$$\dim C^r_d(I) = \begin{cases} d+1 & d < r+1 \\ e(d+1) - v^0(r+1) & d \geq r+1 \end{cases}$$

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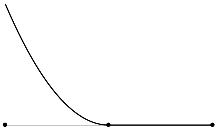

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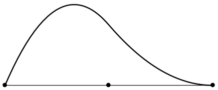

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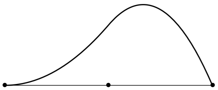

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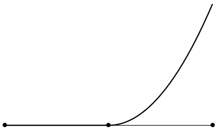

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Dimension Formulas for Triangulations

$\Delta \subset \mathbb{R}^2$ triangulation: f triangles, e^0 interior edges, v^0 interior vertices. For $d \geq 0$,

$$\dim C^0_d(\Delta) = f\frac{(d+2)(d+1)}{2} - e^0(d+1) + v^0$$

In fact, the algebraic structure of $C^0(\Delta)$ is completely combinatorial [Billera '89].

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- $\dim C^r_d(\Delta)$ is known if $d \geq 3r+1$ and Δ is *generic* [Alfeld-Schumaker '90]
- A local basis for $C^r_d(\Delta)$ is known if $d \geq 3r+2$ [Hong '91, Ibrahim-Schumaker '91]

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If Δ is a triangulation in $\mathbb{R}^2$, no known formula for $\dim C^1_3(\Delta)$!

Dimension Formulas for Polygonal Subdivisions

[McDonald-Schenck '09]

$\mathcal{P} \subset \mathbb{R}^2$ a polygonal subdivision (convex polygons): For $d \gg 0$,

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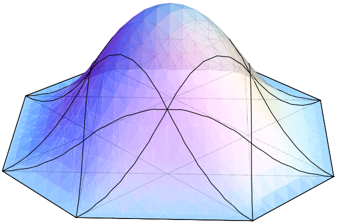
Both results above extend to $C^r_d(\mathcal{P})$, when $\mathcal{P} \subset \mathbb{R}^2$ is planar.

Takeaways

If d is close to $r+1$:

- $\dim C^r_d(\mathcal{P})$ is really hard to compute!
- $C^r_d(\mathcal{P})$ is particularly useful for applications.

THANK YOU!



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