Normal form for codimension two Levi-flat CR singular submanifolds

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Bishop surfaces

$M \subset \mathbb{C}^2$, CR singular submanifold of codimension two with a nondegenerate complex tangent:

$$w = z\bar{z} + \gamma(z^2 + \bar{z}^2) + O(3)$$

if $\gamma \in [0, \infty)$ and for $\gamma = \infty$ we have

$$w = z^2 + \bar{z}^2 + O(3).$$

$\gamma < \frac{1}{2}$ is elliptic, $\gamma = \frac{1}{2}$ is parabolic, and $\gamma > \frac{1}{2}$ is hyperbolic.
If $0 < \gamma < \frac{1}{2}$, Moser-Webster found the normal form

$$w = z\bar{z} + \left(\gamma + \delta(\text{Re } w)^s\right)(z^2 + \bar{z}^2)$$

$\delta = \pm 1, 0$, and $s \in \mathbb{N}$. 


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When $\gamma = 0$,

$$w = z\bar{z}, \quad w = z\bar{z} + z^s + \bar{z}^s + O(s + 1),$$

there are infinitely many formal biholomorphic invariants (Moser, Huang-Krantz, Huang-Yin).
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Various cases of codimension two CR singular manifolds have been studied by Huang-Yin, Dolbeault-Tomassini-Zaitsev, Burcea, Coffman, Slapar, . . .
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How about Levi-flat!

A CR submanifold is Levi-flat if the Levi-form (Levi-map?) vanishes.

It is standard that such a (real-analytic) submanifold is locally

$$\text{Im } z_1 = 0, \quad \text{Im } z_2 = 0.$$  

There are no holomorphic invariants.

In $\mathbb{C}^2$ the notions coincide.
Foliation of Levi-flats

Take $M$ to be

$$\text{Im } z_1 = 0, \quad \text{Im } z_2 = 0.$$ 

$M$ is foliated by complex submanifolds: fix $z_1$ and $z_2$ at some real-value. (The \textit{Levi-foliation})
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The foliation extends (uniquely) to a holomorphic foliation of a neighborhood: leaves are obtained by fixing $z_1$ and $z_2$. 
Our class of submanifolds

Let $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, be a real-analytic CR singular submanifold with a nondegenerate complex tangent at 0, such that $M_{CR}$ is Levi-flat.

We want the normal form for such manifolds.
Our class of submanifolds

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Take $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. Write $M$ as

$$w = \rho(z, \bar{z})$$

for a real-analytic complex-valued function $\rho$ vanishing to second order at the origin. (It is really two real equations).
Detour: mixed-holomorphic submanifolds

For a holomorphic $f$ take $X \subset \mathbb{C}^m$ given by

$$f(z_1, \ldots, z_{m-1}, \bar{z}_m) = 0.$$ 

$X$ is codimension 2. (We can think of it as a complex analytic subvariety, thinking of $\bar{z}_m$ as another holomorphic coordinate, but then our automorphism group is different).
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Exercise, suppose \( m = 2 \): Classify all such submanifolds locally up to local biholomorphisms.
Quadratic parts

In the following let $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, be a germ of a real-analytic real codimension 2 submanifold, CR singular at the origin, written in coordinates $(z, w) \in \mathbb{C}^{n} \times \mathbb{C}$ as

$$w = A(z, \bar{z}) + B(\bar{z}, z) + O(3),$$

for quadratic $A$ and $B$, where $A + B \neq 0$ (nondegenerate complex tangent). Suppose $M$ is Levi-flat (that is $M_{CR}$ is Levi-flat).
Quadratic parts

**Theorem**

(i) *If $M$ is a quadric, then $M$ is locally biholomorphically equivalent to one and exactly one of the following:*

\[
\begin{align*}
(A.1) \quad w &= \bar{z}_1^2, \\
(A.2) \quad w &= \bar{z}_1^2 + \bar{z}_2^2, \\
& \quad \vdots \\
(A.n) \quad w &= \bar{z}_1^2 + \bar{z}_2^2 + \cdots + \bar{z}_n^2, \\
(B.\gamma) \quad w &= |z_1|^2 + \gamma \bar{z}_1^2, \quad \gamma \geq 0, \\
(C.0) \quad w &= \bar{z}_1 z_2, \\
(C.1) \quad w &= \bar{z}_1 z_2 + \bar{z}_1^2.
\end{align*}
\]
Quadratic parts

Theorem

(ii) For general $M$

$$w = A(z, \bar{z}) + B(z, \bar{z}) + O(3)$$

the quadric

$$w = A(z, \bar{z}) + B(z, \bar{z})$$

is Levi-flat, and can be put via a biholomorphic transformation into exactly one of the forms above.
Bishop-like

The quadrics

\begin{align*}
\text{(A.1)} & \quad w = \bar{z}_1^2, \\
\text{(B.}\gamma) & \quad w = |z_1|^2 + \gamma \bar{z}_1^2, \quad \gamma \geq 0.
\end{align*}

These are of the form $N \times \mathbb{C}^{n-1}$ for a Bishop surface $N \subset \mathbb{C}^2$. Not every $M$ with quadratic part of type A.1 or B.\gamma is of the form $N \times \mathbb{C}^{n-1}$.
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If the Levi-foliation of $M$ extends to a non-singular holomorphic foliation of a neighborhood of the origin, then $M$ is either type A.1 or B.\gamma and can be written as $N \times \mathbb{C}^{n-1}$. 
Nondegeneracy

We consider C.1 the “nondegenerate case.”

\[ w = A(z, \bar{z}) + B(\bar{z}, \bar{z}). \]

The form \( A \) “represents the Levi-form.” \( A \) can have rank at most 2 (actually 1) for \( M \) to be Levi-flat.
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For type A.k, the form \( A = 0 \), and so we consider these degenerate. These can be considered as a generalization to higher dimension of Bishop surfaces with Bishop invariant \( \gamma = \infty \).
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For type B.\( \gamma \), the form \( A \) is real-valued and so we also consider it degenerate.

Only C.\( x \) has a complex-valued \( A \), and C.1 also has a nonzero \( B \). These have no analogue in \( \mathbb{C}^2 \).
Stability

Only C.1 and A.\(n\) are stable under perturbation (preserving Levi-flatness, and CR singularity)

CR singularities generally not isolated and can change in type from point to point:

Example:

\[
\omega = \bar{z}_1^2 + \bar{z}_1 z_2 z_3,
\]

is type A.1 at the origin, but of type C.1 at nearby CR singular points.
<table>
<thead>
<tr>
<th>Type</th>
<th>CR singularity $S$</th>
<th>$\dim_{\mathbb{R}} S$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. $k$</td>
<td>$z_1 = 0, \ldots, z_k = 0, \ w = 0$</td>
<td>$2n - 2k$</td>
<td>complex</td>
</tr>
<tr>
<td>B. $\frac{1}{2}$</td>
<td>$z_1 + \bar{z}_1 = 0, \ w = 0$</td>
<td>$2n - 1$</td>
<td>Levi-flat</td>
</tr>
<tr>
<td>B. $\gamma, \gamma \neq \frac{1}{2}$</td>
<td>$z_1 = 0, \ w = 0$</td>
<td>$2n - 2$</td>
<td>complex</td>
</tr>
<tr>
<td>C. 0</td>
<td>$z_2 = 0, \ w = 0$</td>
<td>$2n - 2$</td>
<td>complex</td>
</tr>
<tr>
<td>C. 1</td>
<td>$z_2 + 2\bar{z}_1 = 0, \ w = \frac{-z_2^2}{4}$</td>
<td>$2n - 2$</td>
<td>Levi-flat</td>
</tr>
</tbody>
</table>
Theorem

Suppose $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, is a real-analytic Levi-flat CR singular submanifold of type C.1 or C.0, that is,

$$w = \bar{z}_1 z_2 + \bar{z}_1^2 + O(3) \quad \text{or} \quad w = \bar{z}_1 z_2 + O(3).$$

Then there exists a nonsingular real-analytic foliation defined on $M$ that extends the Levi-foliation on $M_{CR}$, and consequently, there exists a CR real-analytic mapping $F : U \subset \mathbb{R}^2 \times \mathbb{C}^{n-1} \to \mathbb{C}^{n+1}$ such that $F$ is a diffeomorphism onto $F(U) = M \cap U'$, for some neighborhood $U'$ of 0.

Really a Nash blowup, see also a related paper by Garrity.
Note that the Levi-foliation does not always extend (even to $M$ only) for the other types.

Example: A.2:

$$w = \bar{z}_1^2 + \bar{z}_2^2$$

The “leaf” of the foliation becomes singular at the origin.
Mixed-holomorphic C.1

For mixed holomorphic C.1, we completely understand the situation. In this case we can set things up to use implicit function theorem.

**Theorem**

Let $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, be a real-analytic submanifold given by

$$w = \bar{z}_1 z_2 + \bar{z}_1^2 + r(z_1, \bar{z}_1, z_2, z_3, \ldots, z_n),$$

where $r$ is $O(3)$. Then $M$ is Levi-flat and at the origin $M$ is locally biholomorphically equivalent to the quadric $M_{C.1}$ submanifold

$$w = \bar{z}_1 z_2 + \bar{z}_1^2.$$
General normal form for C.1

Could things be really this simple in general?

Theorem

Let $M$ be a real-analytic Levi-flat type C.1 submanifold in $\mathbb{C}^3$. There exists a formal biholomorphic map transforming $M$ into the image of

$^t(\vec{z}; \vec{z}; \vec{w}) = \vec{z} + A(\vec{z}; \vec{w}) \vec{w}$

with $\vec{z} = \vec{z} + 1/2$ and $\vec{w} = \vec{z} + \vec{z}^2$. Here $A = 0$, or $A$ satisfies certain normalizing conditions.

When $A \neq 0$ the formal automorphism group preserving the normal form is finite or 1-dimensional.
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$$
\hat{\phi}(z, \bar{z}, \xi) = \left( z + A(z, \xi, w)w\eta, \xi, w \right)
$$

with $\eta = \bar{z} + \frac{1}{2}\xi$ and $w = \bar{z}\xi + \bar{z}^2$. Here $A = 0$, or $A$ satisfies certain normalizing conditions. When $A \neq 0$ the formal automorphism group preserving the normal form is finite or 1 dimensional.
Automorphisms of the C.1 quadric

Suppose $M \subset \mathbb{C}^3$

$$w = \bar{z}_1z_2 + \bar{z}_1^2,$$

and $(F_1, F_2, G)$ is a local automorphism at the origin, then $F_1$ depends only on $z_1$, $F_2$ and $G$ depend only on $z_2$ and $w$, and $F_1$ completely determines $F_2$ and $G$.

On the other hand, given any $F_1$ with $F_1(0) = 0$, there exist unique $F_2$ and $G$ that complete an automorphism.
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In higher dimensions the extra components of the mapping are arbitrary.
Involution on the C.1 quadric

The proofs use the following key fact:
For the C.1 quadric
\[ w = \bar{z}_1 z_2 + \bar{z}_1^2 \]
we have the involution
\[ (z_1, z_2, \ldots, z_n, w) \mapsto (-\bar{z}_2 - z_1, \ z_2, \ldots, \ z_n, \ w) \]
Thank you!