Normal form for codimension two Levi-flat CR singular submanifolds

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Bishop surfaces

$M \subset \mathbb{C}^2$, CR singular submanifold of codimension two with a nondegenerate complex tangent:

$$w = z\bar{z} + \gamma(z^2 + \bar{z}^2) + O(3)$$

if $\gamma \in [0, \infty)$ and for $\gamma = \infty$ we have

$$w = z^2 + \bar{z}^2 + O(3).$$

$\gamma < \frac{1}{2}$ is elliptic, $\gamma = \frac{1}{2}$ is parabolic, and $\gamma > \frac{1}{2}$ is hyperbolic.

If $0 < \gamma < \frac{1}{2}$, Moser-Webster found the normal form

$$w = z\bar{z} + \left(\gamma + \delta(\operatorname{Re} w)^s\right)(z^2 + \bar{z}^2)$$

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When $\gamma = 0$,

$$w = z\bar{z}, \qquad w = z\bar{z} + z^s + \bar{z}^s + O(s+1),$$

there are infinitely many formal biholomorphic invariants (Moser, Huang-Krantz, Huang-Yin).

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Various cases of codimension two CR singular manifolds have been studied by Huang-Yin, Dolbeault-Tomassini-Zaitsev, Burcea, Coffman, Slapar, ...

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How about Levi-flat!

A CR submanifold is Levi-flat if the Levi-form (Levi-map?) vanishes.

It is standard that such a (real-analytic) submanifold is locally

$$\operatorname{Im} z_1 = 0, \qquad \operatorname{Im} z_2 = 0.$$

There are *no holomorphic invariants*.

In $\mathbb{C}^2$ the notions coincide.

Foliation of Levi-flats

Take M to be

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The foliation extends (uniquely) to a holomorphic foliation of a neighborhood: leaves are obtained by fixing z_1 and z_2.

Our class of submanifolds

Let $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, be a real-analytic CR singular submanifold with a nondegenerate complex tangent at 0, such that M_{CR} is Levi-flat.

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Take $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. Write M as

$$w = \rho(z, \bar{z})$$

for a real-analytic complex-valued function ρ vanishing to second order at the origin. (It is really two real equations).

Detour: mixed-holomorphic submanifolds

For a holomorphic f take $X \subset \mathbb{C}^m$ given by

$$f(z_1, \dots, z_{m-1}, \bar{z}_m) = 0.$$

X is codimension 2. (We can think of it as a complex analytic subvariety, thinking of $\bar{z}_m$ as another holomorphic coordinate, but then our automorphism group is different).

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Exercise, suppose $m = 2$: Classify all such submanifolds locally up to local biholomorphisms.

Quadratic parts

In the following let $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, be a germ of a real-analytic real codimension 2 submanifold, CR singular at the origin, written in coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ as

$$w = A(z, \bar{z}) + B(\bar{z}, \bar{z}) + O(3),$$

for quadratic A and B, where $A + B \not\equiv 0$ (nondegenerate complex tangent). Suppose M is Levi-flat (that is M_{CR} is Levi-flat).

Quadratic parts

Theorem

(i) *If M is a quadric, then M is locally biholomorphically equivalent to one and exactly one of the following:*

$$
\begin{aligned}
&(A.1) && w = \bar{z}_1^2, \\
&(A.2) && w = \bar{z}_1^2 + \bar{z}_2^2, \\
& && \vdots \\
&(A.n) && w = \bar{z}_1^2 + \bar{z}_2^2 + \cdots + \bar{z}_n^2, \\
&(B.\gamma) && w = |z_1|^2 + \gamma \bar{z}_1^2, \quad \gamma \geq 0, \\
&(C.0) && w = \bar{z}_1 z_2, \\
&(C.1) && w = \bar{z}_1 z_2 + \bar{z}_1^2.
\end{aligned}
$$

Quadratic parts

Theorem

(ii) *For general M*

$$w = A(z, \bar{z}) + B(z, \bar{z}) + O(3)$$

the quadric

$$w = A(z, \bar{z}) + B(z, \bar{z})$$

is Levi-flat, and can be put via a biholomorphic transformation into exactly one of the forms above.

Bishop-like

The quadrics

$$(A.1) \quad w = \bar{z}_1^2,$$
$$(B.\gamma) \quad w = |z_1|^2 + \gamma \bar{z}_1^2, \quad \gamma \geq 0.$$

These are of the form $N \times \mathbb{C}^{n-1}$ for a Bishop surface $N \subset \mathbb{C}^2$. Not every M with quadratic part of type A.1 or B.γ is of the form $N \times \mathbb{C}^{n-1}$.

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If the Levi-foliation of M extends to a non-singular holomorphic foliation of a neighborhood of the origin, then M is either type A.1 or B.γ and can be written as $N \times \mathbb{C}^{n-1}$.

Nondegeneracy

We consider C.1 the “nondegenerate case.”

$$w = A(z,\bar{z}) + B(\bar{z},\bar{z}).$$

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Only C.x has a complex-valued A, and C.1 also has a nonzero B. These have no analogue in $\mathbb{C}^2$.

Stability

Only C.1 and A.n are stable under perturbation (preserving Levi-flatness, and CR singularity)

CR singularities generally not isolated and can change in type from point to point:

Example:

$$w = \bar{z}_1^2 + \bar{z}_1 z_2 z_3,$$

is type A.1 at the origin, but of type C.1 at nearby CR singular points.

Quadrics in $\mathbb{C}^{n+1}$

Type	CR singularity S	$\dim_{\mathbb{R}} S$	S
A.k	$z_1 = 0, \ldots, z_k = 0, \; w = 0$	$2n - 2k$	complex
B.$\frac{1}{2}$	$z_1 + \bar{z}_1 = 0, \; w = 0$	$2n - 1$	Levi-flat
B.γ, $\gamma \neq \frac{1}{2}$	$z_1 = 0, \; w = 0$	$2n - 2$	complex
C.0	$z_2 = 0, \; w = 0$	$2n - 2$	complex
C.1	$z_2 + 2\bar{z}_1 = 0, \; w = \frac{-z_2^2}{4}$	$2n - 2$	Levi-flat

Foliations and C.x

Theorem

Suppose $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, is a real-analytic Levi-flat CR singular submanifold of type C.1 or C.0, that is,

$$w = \bar{z}_1 z_2 + \bar{z}_1^2 + O(3) \qquad \text{or} \qquad w = \bar{z}_1 z_2 + O(3).$$

Then there exists a nonsingular real-analytic foliation defined on M that extends the Levi-foliation on M_{CR}, and consequently, there exists a CR real-analytic mapping $F \colon U \subset \mathbb{R}^2 \times \mathbb{C}^{n-1} \to \mathbb{C}^{n+1}$ such that F is a diffeomorphism onto $F(U) = M \cap U'$, for some neighborhood U' of 0.

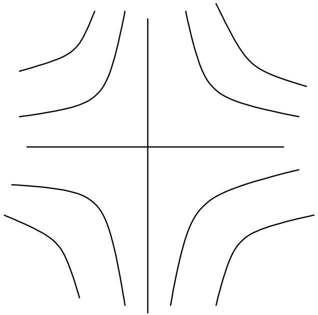
Really a Nash blowup, see also a related paper by Garrity.

Note that the Levi-foliation does not always extend (even to M only) for the other types.

Example: A.2:

$$w = \bar{z}_1^2 + \bar{z}_2^2$$

The “leaf” of the foliation becomes singular at the origin.



Mixed-holomorphic C.1

For mixed holomorphic C.1, we completely understand the situation. In this case we can set things up to use implicit function theorem.

Theorem

Let $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, be a real-analytic submanifold given by

$$w = \bar{z}_1 z_2 + \bar{z}_1^2 + r(z_1, \bar{z}_1, z_2, z_3, \ldots, z_n),$$

where r is $O(3)$. Then M is Levi-flat and at the origin M is locally biholomorphically equivalent to the quadric $M_{C.1}$ submanifold

$$w = \bar{z}_1 z_2 + \bar{z}_1^2.$$

General normal form for C.1

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Theorem

Let M be a real-analytic Levi-flat type C.1 submanifold in $\mathbb{C}^3$. There exists a formal biholomorphic map transforming M into the image of

$$\hat{\varphi}(z, \bar{z}, \xi) = \left(z + A(z, \xi, w)w\eta, \xi, w\right)$$

with $\eta = \bar{z} + \frac{1}{2}\xi$ and $w = \bar{z}\xi + \bar{z}^2$. Here $A = 0$, or A satisfies certain normalizing conditions.
When $A \neq 0$ the formal automorphism group preserving the normal form is finite or 1 dimensional.

Automorphisms of the C.1 quadric

Suppose $M \subset \mathbb{C}^3$

$$w = \bar{z}_1 z_2 + \bar{z}_1^2,$$

and (F_1, F_2, G) is a local automorphism at the origin, then F_1 depends only on z_1, F_2 and G depend only on z_2 and w, and F_1 completely determines F_2 and G.

On the other hand, given any F_1 with $F_1(0) = 0$, there exist unique F_2 and G that complete an automorphism.

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In higher dimensions the extra components of the mapping are arbitrary.

Involution on the C.1 quadric

The proofs use the following key fact:

For the C.1 quadric

$$w = \bar{z}_1 z_2 + \bar{z}_1^2$$

we have the involution

$$(z_1, z_2, \ldots, z_n, w) \mapsto (-\bar{z}_2 - z_1, \quad z_2, \quad \ldots, \quad z_n, \quad w)$$

Thank you!