Polynomials constant on a hyperplane and CR maps of spheres

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Then both Han and I were at another RTG a year later, we got some more ideas.

Proper maps

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For bounded $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^N$ we wish to study holomorphic maps $F\colon U \to V$.

But not just any maps:

A continuous map $F\colon U \to V$ is *proper* if $F^{-1}(K) \subset\subset U$ whenever $K \subset\subset V$.

If F extends continuously to the boundary, i.e. $F\colon \bar{U} \to \bar{V}$, then F is proper if and only if $F(\partial U) \subset \partial V$.

If F is holomorphic and proper, then F is finite-to-one.

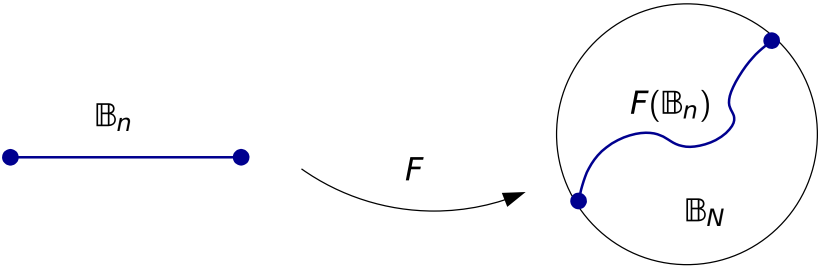
Proper maps of balls

Let

$$\mathbb{B}_n = \text{"unit ball in } \mathbb{C}^n\text{"} = \{z \in \mathbb{C}^n : \|z\| < 1\}.$$

Natural question

For two integers n and N, classify the proper holomorphic maps $F\colon \mathbb{B}_n \to \mathbb{B}_N$.



Exercises

Here are a few exercises:

Exercise

Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc. If $F\colon \mathbb{D} \to \mathbb{D}$ is a proper holomorphic map, then F is a finite Blaschke product. That is, $F(z) = e^{i\theta} \prod_{j=1}^{k} \frac{z-a_j}{1-\bar{a}_j z}$ for some $a_j \in \mathbb{D}$.

Exercise

If $n > N$, no proper holomorphic map $F\colon \mathbb{B}_n \to \mathbb{B}_N$ exists.

Equidimensional case ($n = N$)

When $n = 1$, $z \mapsto z^d$ is a proper map for every $d \in \mathbb{N}$.

Theorem (Alexander '77 (complicated history. . .))

If $F\colon \mathbb{B}_n \to \mathbb{B}_n$ ($n \geq 2$) is a proper holomorphic map then $F \in \operatorname{Aut}(\mathbb{B}_n)$.

An automorphism of the ball is a linear fractional transformation. That is, F can be written as

$$F(z) = U \frac{w - L_w z}{1 - \langle z, w \rangle}$$

for some $w \in \mathbb{C}^n$, a unitary map U, and a linear map L_w.

Note that F is degree 1.

Maps in low dimensions

Theorem (Faran '82)

If $F\colon \mathbb{B}_2 \to \mathbb{B}_3$ is a proper map that is C^3 up to the boundary, then F is spherically equivalent to one of

$$(z_1, z_2) \mapsto (z_1,\ z_2,\ 0) \qquad \textit{(linear embedding)}$$

$$(z_1, z_2) \mapsto (z_1,\ z_1 z_2,\ z_2^2) \qquad \textit{(Whitney map)}$$

$$(z_1, z_2) \mapsto (z_1^2,\ \sqrt{2}\, z_1 z_2,\ z_2^2) \qquad \textit{(deg 2 homogeneous map)}$$

$$(z_1, z_2) \mapsto (z_1^3,\ \sqrt{3}\, z_1 z_2,\ z_2^3) \qquad \textit{(Faran map)}$$

F and G are *spherically equivalent* when there exist automorphisms $\chi \in \operatorname{Aut}(\mathbb{B}_n)$ and $\tau \in \operatorname{Aut}(\mathbb{B}_N)$ such that $F = \tau \circ G \circ \chi$

All (sufficiently smooth) proper maps are rational

Theorem (Forstnerič '89)

Let $N \geq n \geq 2$. If $F : \mathbb{B}_n \to \mathbb{B}_N$ is a proper holomorphic map that is smooth (C^∞) up to the boundary, then F is rational. Furthermore,

$$\deg F \leq C(n, N).$$

Write a rational map $F = f/g$ in lowest terms. Then $\deg F = \max\{\deg f, \deg g\}$.

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Best known bound is:

$$\deg F \leq \frac{N(N-1)}{2(2n-3)}$$

($n = 2$ by Meylan '06, and $n \geq 3$ by D'Angelo-L. '09).

Degree bounds conjecture

Conjecture (D'Angelo)

If $F\colon \mathbb{B}_n \to \mathbb{B}_N$, $n \geq 2$, is a rational proper map then

$$\deg F \leq \begin{cases} 2N-3 & \text{if } n=2, \\ \frac{N-1}{n-1} & \text{if } n \geq 3. \end{cases}$$

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The bound is sharp if true: there exist maps that give equality above.

Getting the sharp bounds for all rational maps is hard.

Known for certain smaller classes, e.g. Huang-Ji-Xu '06 show the sharp bound for maps of "geometric rank 1" for $n \geq 3$.

Similarly true for $n \geq 3$ for polynomial maps constructed by "partial tensoring" (D'Angelo '88).

In this talk we will consider monomial maps.

Monomial maps

A map is *monomial* if every component is a monomial, e.g. $(z_1, z_2) \mapsto (z_1, z_1 z_2, z_2^2)$.

Theorem (D'Angelo-Kos-Riehl '03)

If $F\colon \mathbb{B}_2 \to \mathbb{B}_N$ is a monomial proper map then

$$\deg F \leq 2N - 3.$$

Theorem (L.-Peters '11 and '12?)

If $F\colon \mathbb{B}_n \to \mathbb{B}_N$, $n \geq 3$, is a monomial proper map then

$$\deg F \leq \frac{N-1}{n-1}.$$

Homotopic to monomials?

All known examples of rational proper maps of balls are homotopic to a monomial map in the following sense:

$F \colon \mathbb{B}_n \to \mathbb{B}_N$ and $G \colon \mathbb{B}_n \to \mathbb{B}_N$ are homotopic if there exist a continuous map

$$H \colon \mathbb{B}_n \times [0,1] \to \mathbb{B}_N$$

such that $H(z,0) = F(z)$, $H(z,1) = G(z)$, and $z \mapsto H(z,t)$ is a rational proper map of balls.

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Note: D'Angelo showed that *any* two maps are homotopic if you embed the problem in a sufficiently high dimension.

The real polynomials

Let $f \colon \mathbb{B}_n \to \mathbb{B}_N$ be a monomial proper map, that is:

$$f = \left(c_{\alpha_1} z^{\alpha_1}, c_{\alpha_2} z^{\alpha_2}, \ldots, c_{\alpha_N} z^{\alpha_N}\right)$$

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f takes sphere to sphere so

$$\sum_{j=1}^{N} \left| c_{\alpha_j} z^{\alpha_j} \right|^2 = 1 \qquad \text{on} \quad \sum_{j=1}^{n} \left| z_j \right|^2 = 1$$

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Replace $\left| z_j \right|^2$ with x_j:

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$$\sum_{j=1}^{N} \left|c_{\alpha_j} z^{\alpha_j}\right|^2 = 1 \qquad \text{on} \quad \sum_{j=1}^{n} \left|z_j\right|^2 = 1$$

Replace $\left|z_j\right|^2$ with x_j:

$$p(x) = \sum_{j=1}^{N} \left|c_{\alpha_j}\right|^2 x^{\alpha_j} = 1 \qquad \text{on} \quad \sum_{j=1}^{n} x_j = 1$$

So consider real polynomials $p(x)$ in n variables with nonnegative coefficients such that $p(x) = 1$ on $x_1 + \cdots + x_n = 1$.

N is the number of nonzero coefficients.

The theorem (again)

Theorem (D'Angelo-Kos-Riehl '03, L.-Peters '11 and '12)

Let $p(x_1, \ldots, x_n)$, $n \geq 2$, be a polynomial with nonnegative coefficients such that $p = 1$ whenever $x_1 + \cdots + x_n = 1$. If d is the degree of p and N is the number of coefficients of p, then

$$d \leq \begin{cases} 2N - 3 & \text{if } n = 2, \\ \frac{N-1}{n-1} & \text{if } n \geq 3. \end{cases}$$

Moreover, the inequality is sharp.

Examples

Here's a way to construct examples (e.g. with $n = 3$): We know that $x_1 + x_2 + x_3 = 1$. So start:

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$$\begin{aligned} x_1 + x_2 + x_1x_3(x_1 + x_2 + x_3) + x_2x_3 + x_3^2 &= \\ &= x_1 + x_2 + x_2x_3 + x_3^2 + x_1^2x_3 + x_1x_2x_3 + x_1x_3^2 \end{aligned}$$

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$$= x_1 + x_2 + x_2x_3 + x_3^2 + x_1^2x_3 + x_1x_2x_3 + x_1x_3^2$$

We'll call such polynomials generalized Whitney polynomials.

Exercise

For those that are bored (and haven't seen this before)

Exercise

Easy: Show that the polynomial corresponding to the Faran map $p(x_1, x_2) = x_1^3 + 3x_1x_2 + x_2^3$ *is not generalized Whitney.*

Harder: Show how you can obtain this polynomial if you also allow dividing $(x_1 + x_2)$*.*

The quotient

If $p(x) = 1$ whenever $x_1 + \cdots + x_n = 1$, then let

$$q(x) = \frac{p(x) - 1}{x_1 + \cdots + x_n - 1}$$

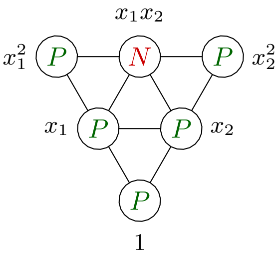
For example: if $p(x_1, x_2) = x_1^3 + 3x_1x_2 + x_2^3$ then

$$q(x) = \frac{x_1^3 + 3x_1x_2 + x_2^3 - 1}{x_1 + x_2 - 1} = x_1^2 - x_1x_2 + x_2^2 + x_1 + x_2 + 1.$$

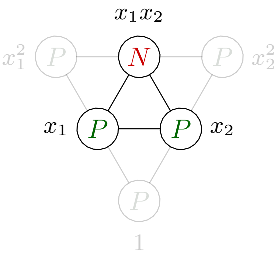
Newton diagram ($n = 2$)

Now take $q(x)$ and write down its Newton diagram also marking positive and negative coefficients. For example,

$$\frac{x_1^3 + 3x_1x_2 + x_2^3 - 1}{x_1 + x_2 - 1} = +x_1^2 - x_1x_2 + x_2^2 + x_1 + x_2 + 1.$$



Sinks



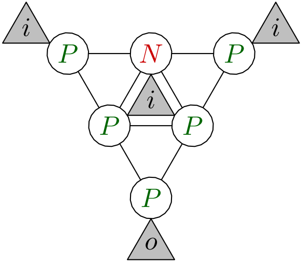
$$(x_1 + x_2 - 1)(Px_1 + Px_2 + Nx_1x_2) =$$

$$= (P + P - N)x_1x_2 + Px_1^2 + Nx_1^2x_2 + Px_2^2 + Nx_1x_2^2 - Px_1 - Px_2$$

This is the only way that x_1x_2 is formed in $p - 1$ so there must be x_1x_2 term in $p - 1$. Call this a *sink*. If a negative term is forced, we say it is a *source*.

All sinks sources

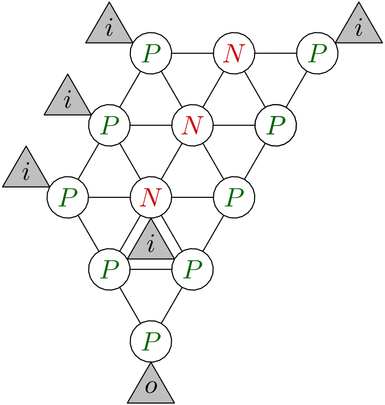
So here they all are (i for sink, o for source):



So in $p-1$, there are positive terms x_1^3, x_1x_2, and x_2^3, and there is a negative term 1.

Remember $p = x_1^3 + 3x_1x_2 + x_2^3$

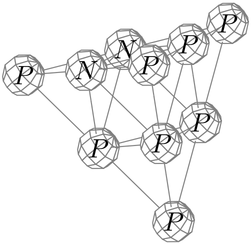
Now do it for a complicated diagram



And obtain $d \leq 2N - 3$, where d is the "size" of the diagram, N is the number of sinks, and there should be at most one source. (Here $d = 5$, $N = 5$, so $5 \leq 2(5) - 3 = 7$)

In higher dimensions

We get higher dimensional newton diagrams. For $n = 3$ we get 3 dimensional diagrams.



This is the diagram for

$$p(x_1, x_2, x_3) = x_1^3 + 3x_1(x_2 + x_3) + (x_2 + x_3)^3,$$

which has 7 terms.

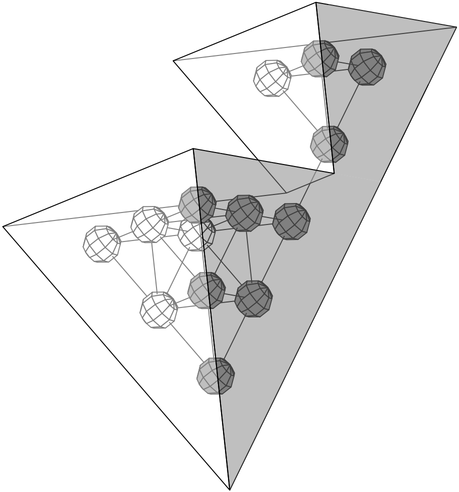
Throw away the insides

1) Consider the diagram as a solid and look at its faces.

2) Now apply the 2-dimensional result to each face.

3) Obtain:

$$d \le \frac{N-1}{2}$$



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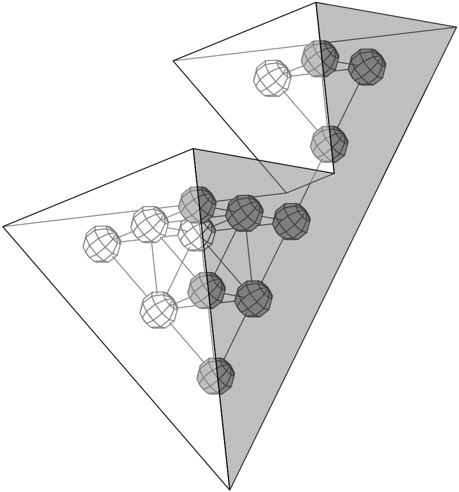
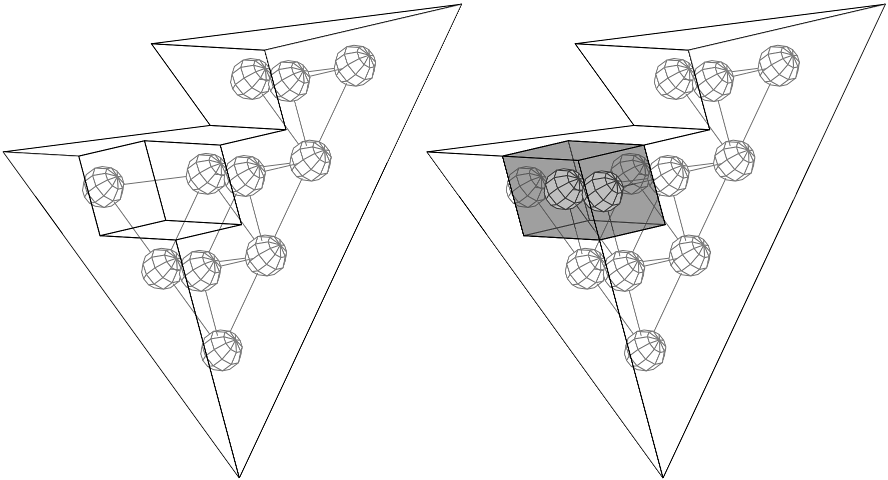
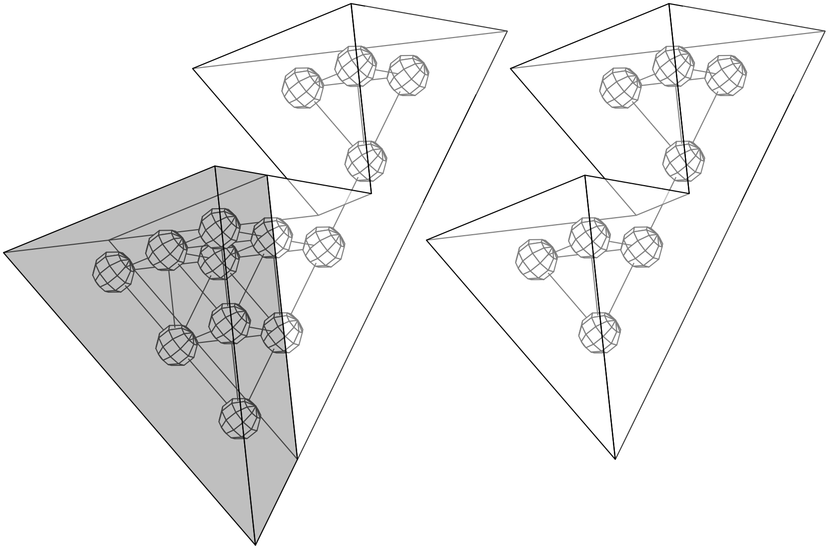
OK this is possible for "nice" diagrams, but not easy for "ugly" diagrams.



So 1) fill



2) cut



3) Induction!

If it were only so easy

There are of course some technicalities.

1) You have to allow sources to appear.

2) You have to handle “inside” edges of two dimensional diagrams.

3) The top faces are slightly different.

Now finally obtain:

$$d \leq \frac{N-1}{2}$$

4 dimensions are hard to draw

In 4 dimensions we cannot fill, but ...

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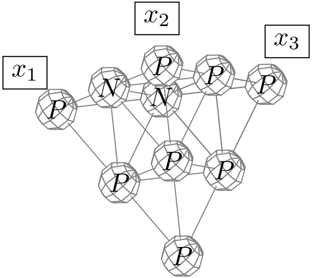
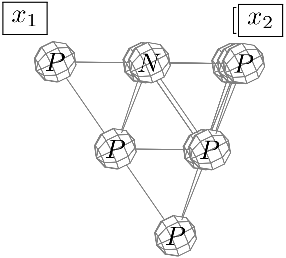
We can prove the estimate for $n \geq 4$ by induction starting with $n = 3$.

What we do is "view" the diagram along an edge. Here is an example from 3 to 2 dimensions (I can't draw 4 dimensions)





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The “view” of a 4 dimensional diagram is a 3 dimensional diagram.

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It is not possible to start at $n=2$ to get the sharp bound. You must use $n=3$ (the hardest case).

Classification of sharp polynomials ($n \geq 4$)

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So if we obtain equality in the bound, that is $N = d(n-1)+1$, the polynomial is a generalized Whitney polynomial.

Recall: a generalized Whitney polynomial is obtained by multiplication using $x_1 + \cdots + x_n$, e.g.

$$x_1 + x_2 + x_3 + x_4\bigl(x_1 + x_2 + x_3 + x_4(x_1 + x_2 + x_3 + x_4)\bigr)$$

Sharp polynomials in $n = 3$

For $n = 3$ there are sharp generalized Whitney polynomials.
e.g.

$$x_1 + x_2 + x_3(x_1 + x_2 + x_3(x_1 + x_2 + x_3))$$

has 7 terms (and $\frac{N-1}{2} = \frac{7-1}{2} = 3 = d$)

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But, the polynomial

$$p(x_1, x_2, x_3) = x_1^3 + 3x_1(x_2 + x_3) + (x_2 + x_3)^3,$$

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We don't know if other non-Whitney sharp examples exist, if you exclude ones formed using p and $x_1 + x_2 + x_3$ such as:

$$x_1 + x_2 + x_3(x_1^3 + 3x_1(x_2 + x_3) + (x_2 + x_3)^3)$$

Sharp polynomials in $n = 2$

When $n = 2$, no sharp polynomial of degree 3 or greater is Whitney. Here are all the sharp polynomials up to degree 17. (Lichtblau,L. '10)

d	Sharp polynomials, $p(x,y) = 1$ when $x + y = 1$
1	$x + y$
3	$x^3 + 3xy + y^3$
5	$x^5 + 5x^3y + 5xy^2 + y^5$
7	$x^7 + 7x^3y + 14x^2y^3 + 7xy^5 + y^7$
	$x^7 + 7x^3y + 7x^3y^3 + 7xy^3 + y^7$
	$x^7 + \frac{7}{2}x^5y + \frac{7}{2}xy + \frac{7}{2}xy^5 + y^7$
9	$x^9 + 9x^7y + 27x^5y^2 + 30x^3y^3 + 9xy^4 + y^9$
11	$x^{11} + 11x^9y + 44x^7y^2 + 77x^5y^3 + 55x^3y^4 + 11xy^5 + y^{11}$
	$x^{11} + 11x^5y + 11x^5y^5 + 55x^4y^3 + 55x^3y^5 + 11xy^5 + y^{11}$
13	$x^{13} + 13x^{11}y + 65x^9y^2 + 156x^7y^3 + 182x^5y^4 + 91x^3y^5 + 13xy^6 + y^{13}$
	$x^{13} + 13x^{11}y + 65x^9y^2 + \frac{221}{2}x^7y^3 + \frac{92}{2}x^3y^3 + \frac{91}{2}x^3y^7 + 13xy^6 + y^{13}$
	$x^{13} + \frac{234}{25}x^{11}y + \frac{143}{5}x^8y^2 + \frac{143}{5}x^7y^4 + \frac{91}{25}xy + \frac{143}{25}xy^6 + \frac{91}{25}xy^{11} + y^{13}$
	$x^{13} + \frac{234}{25}x^{11}y + \frac{143}{5}x^9y^2 + \frac{143}{5}x^7y^3 + \frac{91}{25}xy + \frac{143}{25}xy^6 + \frac{91}{25}xy^{11} + y^{13}$
15	$x^{15} + 15x^{13}y + 90x^{11}y^2 + 275x^9y^3 + 450x^7y^4 + 378x^5y^5 + 140x^3y^6 + 15xy^7 + y^{15}$
	$x^{15} + 140x^9y^3 + 15x^7y + 420x^7y^4 + 15x^7y^7 + 378x^5y^5 + 140x^3y^6 + 15xy^7 + y^{15}$
17	$x^{17} + 17x^{15}y + 119x^{13}y^2 + 442x^{11}y^3 + 935x^9y^4 + 1122x^7y^5 + 714x^5y^6 + 204x^3y^7 + 17xy^8 + y^{17}$

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Let $p(x) = q(x)(x_1 + \cdots + x_n)^d$ *be a nonzero polynomial, then* $p(x)$ *has at least* $\binom{n-1+d}{d}$ *terms.*

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Which can be used to prove

Corollary (D'Angelo-L. '11?)

If $\sum_{j=1}^{N} \pm \left|f_j(z)\right|^2 = q(z,\bar{z}) \, \|z\|^{2d}$ is nonzero, then $N \geq \binom{n-1+d}{d}$.

Further directions II...

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Suppose no monomial divides $p(x)$, and if

$$p(x) = p_1(x)(x_1 + \cdots + x_n) + p_2(x)(x_1 + \cdots + x_n)$$

where p_1 and p_2 have no monomials in common, then p_1 or p_2 is zero. Call p *indecomposable*.

Theorem (L.,-Peters '11)

If N is the number of terms in p and p is indecomposable, then

$$\deg p \leq C(n, N)$$

We only know the sharp value of C for $n = 3$.

Examples

$$x_1^2 + x_1x_2 + x_3(-x_2 - x_3) = x_1^2 + x_1x_2 - x_2x_3 - x_3^2$$

is indecomposable.

$$x_1 + x_2 + x_3 + x_1^{d-1}(x_1 + x_2 + x_3) = x_1 + x_2 + x_3 + x_1^d + x_1^{d-1}x_2 + x_1^{d-1}x_3$$

is decomposable, has 6 terms and has arbitrary degree d.

How to use the result

$$Q(a,b) \overset{def}{=} \left\{ z \in \mathbb{C}^n \ : \ \sum_{j=1}^{a} \left|z_j\right|^2 - \sum_{j=a+1}^{a+b} \left|z_j\right|^2 = 1 \right\}$$

Say a rational map $f\colon Q(a,b) \to Q(A,B)$ is indecomposable "if we cannot write it in homogeneous coordinates as a direct sum of two hyperquadric maps."

Corollary (L., Peters '11)

Let $n = a + b \geq 2$ and $N = A + B$. Let $f\colon Q(a,b) \to Q(A,B)$ be an indecomposable (rational) monomial map. Then

$$\deg f \leq C(n,N)$$

For monomial maps indecomposable translates to the condition of the previous slide.