Hermitian forms and rational maps of hyperquadrics

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Hyperquadrics

Define

$$Q(a,b) \overset{def}{=} \left\{ z \in \mathbb{C}^n \;\middle|\; \sum_{j=1}^{a} \left|z_j\right|^2 - \sum_{j=a+1}^{a+b} \left|z_j\right|^2 = 1 \right\}$$

Note that

$$S^{2n-1} = Q(n,0)$$

CR maps

A map $F\colon Q(a,b) \to Q(c,d)$ is CR if it is continuously differentiable and satisfies the tangential Cauchy-Riemann equations. A real-analytic CR map is a restriction of a holomorphic map.

Natural question

Classify all CR mappings of $Q(a,b)$ to $Q(c,d)$.

Q-equivalence

Let $\mathrm{Aut}(Q(a,b))$ denote the set of linear fractional transformations preserving $Q(a,b)$.

$F\colon Q(a,b) \to Q(c,d)$ and $G\colon Q(a,b) \to Q(c,d)$ are *Q-equivalent* if there exist linear fractional maps $\chi \in \mathrm{Aut}(Q(a,b))$ and $\tau \in \mathrm{Aut}(Q(a,b))$ such that the following diagram commutes:

$$\begin{tikzcd} Q(a,b) \arrow[r, "F"] \arrow[d, "\chi"] & Q(c,d) \arrow[d, "\tau"] \\ Q(a,b) \arrow[r, "G"] & Q(c,d) \end{tikzcd}$$

When both hyperquadrics are spheres, then Q-equivalence is called *spherical equivalence*.

Q-equivalence

Let $F\colon Q(a,b) \to Q(c_1,d_1)$ and $G\colon Q(a,b) \to Q(c_2,d_2)$ be CR maps.

F and G are Q-equivalent if there exists a linear fractional map $L\colon Q(c_1,d_1) \to Q(c_2,d_2)$ and $L \circ F$ is Q-equivalent to G.

(or vice versa)

Note: $Q(a,b) \cong Q(b+1,a-1)$ by a linear fractional mapping.

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$$z \mapsto e^{i\theta} \prod_{j=1}^{k} \frac{z - a_j}{1 - \bar{a}_j z}$$

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- In $\mathbb{C}^2$ we need only consider S^3 (as $Q(1,1) \cong Q(2,0)$).
- No CR maps from S^3 to S^1.
- The only CR maps of S^3 to S^3 are automorphisms by a theorem of Pinčuk (also others ...)

$\mathbb{C}^2$ to $\mathbb{C}^3$

The only hyperquadric in $\mathbb{C}^2$ is S^3.

In $\mathbb{C}^3$ there are two hyperquadrics S^5 and $Q(2,1)$.

Theorem (Faran '82)

Let $U \subset S^3$ be connected and open. Let $F \colon U \to S^5$ be a nonconstant C^3 CR map. Then F is spherically equivalent to exactly one of

(i) $(z,w) \mapsto (z,\ w,\ 0)$ *("Linear")*

(ii) $(z,w) \mapsto (z,\ zw,\ w^2)$ *("Whitney map")*

(iii) $(z,w) \mapsto (z^2,\ \sqrt{2}\, zw,\ w^2)$ *("Homogeneous")*

(iv) $(z,w) \mapsto (z^3,\ \sqrt{3}\, zw,\ w^3)$ *("Faran map")*

Theorem (— '09)

Let $U \subset Q(2,0)$ be connected and open. Let $F\colon U \to Q(2,1)$ be a nonconstant real-analytic CR map. Then F is Q-equivalent to exactly one of:

(i) $(z,w) \mapsto (z,w,0)$,

(ii) $(z,w) \mapsto (z^2, \sqrt{2}\,w, w^2)$,

(iii) $(z,w) \mapsto \left(\frac{1}{z}, \frac{w^2}{z^2}, \frac{w}{z^2}\right)$,

(iv) $(z,w) \mapsto \left(\frac{z^2+\sqrt{3}\,zw+w^2-z}{w^2+z+\sqrt{3}\,w-1}, \frac{w^2+z-\sqrt{3}\,w-1}{w^2+z+\sqrt{3}\,w-1}, \frac{z^2-\sqrt{3}\,zw+w^2-z}{w^2+z+\sqrt{3}\,w-1}\right)$,

(v) $(z,w) \mapsto \left(\frac{\sqrt[4]{2}(zw-iz)}{w^2+\sqrt{2}\,iw+1}, \frac{w^2-\sqrt{2}\,iw+1}{w^2+\sqrt{2}\,iw+1}, \frac{\sqrt[4]{2}(zw+iz)}{w^2+\sqrt{2}\,iw+1}\right)$,

(vi) $(z,w) \mapsto \left(\frac{2w^3}{3z^2+1}, \frac{z^3+3z}{3z^2+1}, \sqrt{3}\,\frac{wz^2-w}{3z^2+1}\right)$,

(vii) $(z,w) \mapsto (1, g(z,w), g(z,w))$ *for an arbitrary CR function g.*

Degree 2 maps

Theorem (— '09)

Let $F\colon S^{2n-1} \to S^{2N-1}$, $n \geq 2$, be a rational CR mapping with $\deg F \leq 2$. Then F is is spherically equivalent to a monomial map (every component is a monomial).

In particular, f is equivalent to a map taking $(z_1, \ldots, z_n)$ to

$$\Big(\sqrt{t_1}\,z_1, \sqrt{t_2}\,z_2, \ldots, \sqrt{t_n}\,z_n, \sqrt{1-t_1}\,z_1^2, \sqrt{1-t_2}\,z_2^2, \ldots, \sqrt{1-t_n}\,z_n^2, \\ \sqrt{1-t_1-t_2}\,z_1 z_2, \sqrt{1-t_1-t_3}\,z_1 z_3, \ldots, \sqrt{1-t_{n-1}-t_n}\,z_{n-1} z_n\Big)$$

for $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq 1$.

Furthermore, all maps of the above form are mutually spherically inequivalent for different parameters $(t_1, \ldots, t_n)$.

The moduli space is a closed n-simplex.

In particular, every equivalence class can be represented by

$$Lz \oplus \left((\sqrt{I - L^* L}\, z) \otimes z \right)$$

where L is a diagonal matrix with entries in the interval $[0, 1]$ and sorted by size.

The theorem is optimal. The conclusion is not true if $n = 1$, and it is not true if $\deg F \geq 3$.

Previous work

For $n = 2$, a classification of proper maps of degree 2 was known previously by a paper of Ji and Zhang ('09).

Previously, Faran, Huang, Ji, and Zhang ('10) have shown that for $n = 2$, all degree two maps are equivalent to a polynomial. They also construct a degree 3 mapping which is not equivalent to a polynomial mapping.

Real polynomials

Let $p(x, y)$ be a real polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$ (or $\mathbb{C}^n$ where $z = x + iy$).

Define the polynomial r on $\mathbb{C}^n \times \mathbb{C}^n$ by

$$r(z, \bar{w}) = p\left(\frac{z + \bar{w}}{2}, \frac{z - \bar{w}}{2i}\right).$$

So $r(z, \bar{z}) = p(x, y)$ and r is *Hermitian symmetric* in the sense that $r(z, \bar{w}) = \overline{r(w, \bar{z})}$.

Let's bihomogenize

It is better at this point to work in projective space $\mathbb{CP}^n$ (the space of complex lines through the origin in $\mathbb{C}^{n+1}$). From now on let

$$z = (z_1, z_2, \ldots, z_{n+1})$$

be the homogeneous coordinates on $\mathbb{CP}^n$.

We assume will assume that r is *bihomogeneous* of *bidegree* (d, d), that is

$$t^d r(z, \bar{z}) = r(tz, \bar{z}) = r(z, t\bar{z}).$$

The matrix of coefficients

Write r in multi-index notation

$$r(z,\bar{w}) = \sum_{\alpha,\beta} c_{\alpha\beta} z^\alpha \bar{w}^\beta.$$

The matrix $C = [c_{\alpha\beta}]$ is Hermitian symmetric.

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The matrix $C = [c_{\alpha\beta}]$ is Hermitian symmetric.

If C has a positive and b negative eigenvalues. Write

$$C = \sum_{j=1}^{a} v_j v_j^* - \sum_{j=a+1}^{a+b} v_j v_j^*$$

for some vectors v_j (for example eigenvectors of C).

The decomposition

Suppose r is of bidegree (d, d). Let $\mathcal{Z}$ be the Veronese mapping of degree d, that is the vector of all monomials $\mathcal{Z} = (\dots, z^\alpha, \dots)^t$.

$$r(z, \bar{z}) = \langle C\mathcal{Z}, \mathcal{Z} \rangle = \mathcal{Z}^* C \mathcal{Z} = \sum_{j=1}^{a} \mathcal{Z}^* v_j v_j^* \mathcal{Z} - \sum_{j=a+1}^{a+b} \mathcal{Z}^* v_j v_j^* \mathcal{Z} = \sum_{j=1}^{a} \left|\phi_j(z)\right|^2 - \sum_{j=a+1}^{a+b} \left|\phi_j(z)\right|^2,$$

where $\phi_j(z)$ is the polynomial $v_j^* \mathcal{Z}$.

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$$\sum_{j=1}^{a} \mathcal{Z}^* v_j v_j^* \mathcal{Z} - \sum_{j=a+1}^{a+b} \mathcal{Z}^* v_j v_j^* \mathcal{Z} =$$

$$\sum_{j=1}^{a} \left|\phi_j(z)\right|^2 - \sum_{j=a+1}^{a+b} \left|\phi_j(z)\right|^2,$$

where $\phi_j(z)$ is the polynomial $v_j^* \mathcal{Z}$.

If we let $f = (\phi_1, \dots, \phi_a)$ and $g = (\phi_{a+1}, \dots, \phi_{a+b})$. Then $r(z, \bar{z}) = \left\|f(z)\right\|^2 - \left\|g(z)\right\|^2$.

Hyperquadrics in $\mathbb{CP}^n$

Suppose that $a + b = n + 1$. Let $\tau \colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ be the standard projection. Define

$$HQ(a,b) \overset{def}{=} \left\{ \tau(z) \in \mathbb{CP}^n \;\middle|\; \sum_{j=1}^{a} \left|z_j\right|^2 - \sum_{j=a+1}^{a+b} \left|z_j\right|^2 = 0 \right\}$$

Note that $Q(a,b)$ is the dehomogenized $HQ(a,b+1)$.

Therefore the sphere S^{2n-1} is $HQ(n,1)$.

Hyperquadrics in $\mathbb{CP}^n$

The defining equation for $HQ(a,b)$ is

$$|z_1|^2 + \cdots + |z_a|^2 - |z_{a+1}|^2 - \cdots - |z_{a+b}|^2 = 0$$

Let $V = V_{a,b}$ be the $n+1$ by $n+1$ matrix

$$\begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & \dots & 0 & -1 \end{bmatrix}$$

with a ones and b negative ones on the diagonal.

Then $HQ(a,b)$ is defined by $\langle V_{a,b} z, z \rangle = 0$.

Hermitian forms and CR maps of hyperquadrics

Suppose r vanishes on $HQ(a,b)$, i.e.

$$r(z,\bar{z}) = q(z,\bar{z})\langle V_{a,b}z, z\rangle.$$

Hermitian forms and CR maps of hyperquadrics

Suppose r vanishes on $HQ(a, b)$, i.e.

$$r(z, \bar{z}) = q(z, \bar{z}) \langle V_{a,b} z, z \rangle .$$

If the Hermitian form of r has signature (c, d). Write

$$r(z, \bar{z}) = \left\| f(z) \right\|^2 - \left\| g(z) \right\|^2 .$$

The map $z \mapsto (f(z), g(z))$ takes $HQ(a, b)$ to $HQ(c, d)$.

Hermitian forms and CR maps of hyperquadrics

Suppose r vanishes on $HQ(a,b)$, i.e.

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If the Hermitian form of r has signature (c,d). Write

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The map $z \mapsto (f(z), g(z))$ takes $HQ(a,b)$ to $HQ(c,d)$.

Conversely if we have a map $F \colon HQ(a,b) \to HQ(c,d)$ then let $r(z,\bar{z}) = \langle V_{a,b}\, F(z), F(z)\rangle$.

Linear independence

It is enough to consider maps with linearly independent components.

If $F \colon HQ(a,b) \to HQ(c,d)$ does not have linearly independent components, then after a linear change of coordinates on the target we have a map from $HQ(a,b)$ to $HQ(c',d') \times \mathbb{C}^k$ for some $c' \leq c$, $d' \leq d$ ($c' + d' < c + d$) and $k \geq 1$.

Key idea

Lemma

Let H be a nonsingular Hermitian matrix. Let F and G be homogeneous polynomial maps of $\mathbb{C}^{n+1}$ to $\mathbb{C}^{N+1}$ such that

$$\langle H F(z), F(z) \rangle = \langle H G(z), G(z) \rangle.$$

Suppose that F has linearly independent components. Then

$$G(z) = L F(z)$$

for some invertible matrix L such that $L^* H L = H$.

In other words, if two maps F and G of hyperquadrics (with linearly independent components) give the same Hermitian form, then they differ by an automorphism of the target.

Let us repeat

To classify maps from $HQ(a,b)$ to $HQ(c,d)$ up to Q-equivalence, it is enough to classify (up to $\mathrm{Aut}(HQ(a,b))$) real polynomials of signature pair (c,d) that vanish on $HQ(a,b)$

We have removed the automorphism group of the target from the classification problem!

Degree 2 maps

Degree 2 maps correspond to bidegree (2,2) bihomogeneous polynomials vanishing on $HQ(a,b)$

Write

$$r(z,\bar{z}) = \langle Az, z\rangle \langle V_{a,b}z, z\rangle.$$

An element of $\operatorname{Aut}(HQ(a,b))$ is represented by a matrix T such that $T^* V_{a,b} T = J$. Hence,

$$r(Tz,\overline{Tz}) = \langle ATz, Tz\rangle \langle V_{a,b}Tz, Tz\rangle = \\ \langle T^*ATz, z\rangle \langle T^*V_{a,b}Tz, z\rangle = \langle T^*ATz, z\rangle \langle V_{a,b}z, z\rangle.$$

The classification

Therefore to classify degree 2 maps, we simply need to find a normal form for the pair of Hermitian matrices $(A, V_{a,b})$ under $*$-congruence.

Done in the 1930s almost independently by Trott, Turnbull, Ingraham and Wegner, and Williamson. It was rediscovered later many times.

The normal form for the matrix pair gives all normal forms for all degree 2 maps between all hyperquadrics.

Calculation required now.

For example, we can show that the only normal forms that correspond to sphere maps are diagonal ones (hence monomial).

Rigidity for hyperquadrics

Let $U \subset HQ(a,b)$ be connected and open. Let $F \colon U \to HQ(c,d)$ be a real analytic CR map with linearly independent components. Assume $2 \leq b \leq a$, $c \geq a$, and $d \geq b$.

Theorem (Baouendi-Huang '05)

If $d = b$ then F is Q-equivalent to a linear embedding (and hence $c = a$).

Theorem (Baouendi-Ebenfelt-Huang '09)

If $d < 2b - 1$ then F is Q-equivalent to a linear embedding (and hence $c = a$ and $d = b$).

Stability for spheres

There are lots of sphere maps.

Theorem (D'Angelo, — '08)

For every $N \geq n^2 - 2n + 2$ *there exists a CR map* $F : HQ(n,1) \to HQ(N,1)$ *with linearly independent components.*

That is, there exist CR (polynomial) maps of S^{2n-1} to S^{2N-1} where N is the minimal embedding dimension.

More stability

Theorem (D'Angelo, — '10)

Given $n \geq 1$, there exists an integer M such that for any pair (A, B) with $A + B \geq M$ and $A, B \geq 2$, there exist rational mappings from $HQ(n, 1)$ to $HQ(A, B)$ with linearly independent components and whose degrees can be chosen arbitrarily large.

Maximal degrees for $n = 2$

Maximal degrees for maps of $HQ(2,1)$ to $HQ(A,B)$ with linearly independent components.

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
5	-	f	∞	∞	∞	∞	$\cdots$
4	-	f	∞	∞	∞	∞	$\cdots$
3	-	3	e	∞	∞	∞	$\cdots$
2	-	1	3	e	∞	∞	$\cdots$
1	-	-	1	3	f	f	$\cdots$
0	0	-	-	-	-	-	$\cdots$
$^B/_A$	0	1	2	3	4	5	$\cdots$

Construction of unbounded maps

Let $F\colon HQ(a,b) \to HQ(c,d)$ is a CR map with linearly independent components. Let g be an arbitrary CR function on $HQ(a,b)$.

Let $r(z,\bar{z})$ be the Hermitian form representing F. Then write

$$\tilde{r}(z,\bar{z}) = r(z,\bar{z}) \pm \left|g(z)\right|^2 \langle V_{a,b} z, z \rangle$$

The mapping $\tilde{F}$ corresponding to $\tilde{r}$ takes $HQ(a,b)$ to $HQ(c+a, d+b)$ or $HQ(c+b, d+a)$.

Proof of stability

Suppose there exists a map from $HQ(n,1)$ to $HQ(N,1)$ (lin. ind. components). For example, $N \geq n^2 - 2n + 2$.

We have unbounded maps of $HQ(n,1)$ to $HQ(A,B)$ with lin. ind. components for all A, B:

$$(A,B) = (N,1) + a(n,1) + b(1,n),$$

$$(A,B) = (1,N) + a(n,1) + b(1,n).$$

($a, b \geq 0$ and not both zero)

Let $M = 2(2n^2 - n)$. Now if $A, B \geq 2$ and $A + B \geq M$, we get an unbounded map.