RESEARCH STATEMENT

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1. Introduction and research philosophy

My primary interests lie in complex analysis in general and CR geometry in particular. My research in CR geometry has also led me to study problems in real and complex algebraic geometry, differential equations, discrete geometry, combinatorics, number theory, and experimental mathematics using computers. My research philosophy is not simply to solve problems within the confines of a particular area, but to look for connections and applications to other areas of mathematics and even other disciplines.

In CR geometry, I am primarily interested in the study of singularities and complexity. A real object in a complex manifold inherits a certain amount of the complex structure, called the CR structure, from the ambient manifold. CR geometry is the study of these objects and maps preserving the CR structure. CR complexity involves the study of CR maps from a manifold to a sphere or hyperquadric; it is the analogue of the Nash embedding in CR geometry where the role of flat space is played by spheres and hyperquadrics. We ask which maps exist, and for those maps we ask how complicated they are. A related natural question is to understand the CR geometry of singular objects, or the singularities of the CR structure itself. Many fundamental questions are not yet fully answered and offer a rich array of topics for future study.

Consider singular Levi-flat hypersurfaces and CR maps between spheres and hyperquadrics. The common link is the equation $\|f(z)\|^2 - \|g(z)\|^2 = 0$ for holomorphic maps f and g. When g is scalar-valued, f/g maps the solution set to the sphere. If f and g are scalar-valued the solution set is Levi-flat, a set with minimal CR complexity. The study of CR maps between spheres has a rich combinatorial and discrete geometric aspect, lending itself naturally to computer experimentation.

2. CR maps between spheres and hyperquadrics

If $M \subset \mathbb{C}^n$ and $M' \subset \mathbb{C}^N$ are real submanifolds, then a map $\varphi\colon M \to M'$ is CR if it satisfies the tangential Cauchy-Riemann equations, for example the restriction of a holomorphic map. A hard fundamental question is to classify CR maps.

Let M' be a hyperquadric, that is, M' is defined by $\langle z, z\rangle = 1$, where $\langle \cdot, \cdot \rangle$ is a nondegenerate (but not necessarily positive definite) Hermitian product. For such M', the classification of the CR maps $\varphi\colon M \to M'$ amounts to understanding the ideal of real functions vanishing on M. For simplicity, let $\rho(z, \bar{z})$ be a polynomial vanishing on M. Write

$$\rho(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2 , \tag{1}$$

where f and g are holomorphic maps to some finite-dimensional space. The target dimension of the map f (resp. g) is the number of positive (resp. negative) eigenvalues of the matrix of coefficients of ρ. The map (f, g) induces a CR map $\varphi\colon M \to M'$, unique up to fractional linear transformations preserving M'. The problem rests in studying the signature pair (the number of positive and negative eigenvalues) of functions in the ideal generated by M.

A well-studied case of the mapping problem is when $M = S^{2n-1} \subset \mathbb{C}^n$ and $M' = S^{2N-1} \subset \mathbb{C}^N$ are unit spheres. When $N < n$ no nonconstant CR maps exist. When $n = N = 1$ the map z^d takes the unit circle to itself and is of degree d, which is arbitrary. On the other hand a well-known theorem (by Pincuk, Alexander, and others) states that if $n = N \geq 2$, then any CR map of spheres must be linear fractional, a rational map of degree 1.

Two maps are *spherically equivalent* if the they are conjugates of each other using automorphisms of the sphere. Degree-one maps are equivalent to the identity. A map is *monomial* if each component is a single monomial.

Theorem 2.1 (Lebl [19])**.** *Let $f\colon S^{2n-1} \to S^{2N-1}$, $n \geq 2$, be a rational CR map of degree 2. Then f is spherically equivalent to a monomial map.*

Furthermore, the normal form for such a mapping is a map that takes $z \in \mathbb{C}^n$ to

$$Lz \oplus (\sqrt{I - L^*L}z) \otimes z, \tag{2}$$

where L is a diagonal matrix with nonnegative diagonal entries sorted by size, such that $I - L^*L$ also has nonnegative entries. All maps of the form (2) *are mutually spherically inequivalent.*

Forstnerič [11] proved that if $n \geq 2$ and the map is C^∞ up to the boundary, then the map is rational of degree bounded by a function of n and N only. A sharp bound on the degree is unknown. D'Angelo conjectured that the degree d of the map satisfies the sharp bound:

$$d \leq \begin{cases} 2N - 3 & \text{if } n = 2, \\ \frac{N-1}{n-1} & \text{if } n \geq 3. \end{cases} \tag{3}$$

Monomial examples that achieve equality exist. The best currently known bound that applies to all rational proper maps was proved by Meylan [30] for $n = 2$ and extended to $n \geq 3$ by D'Angelo and myself [6], and it is $d \leq \frac{N(N-1)}{2(2n-3)}$.

The combinatorics in the monomial case most likely captures the complexity of the general problem. For monomial maps we have:

Theorem 2.2 (D'Angelo-Kos-Riehl [4] for $n = 2$, and Lebl-Peters [27,28] for $n \geq 3$)**.** *Suppose that $f\colon S^{2n-1} \to S^{2N-1}$, $n \geq 2$, is a monomial CR map of degree d. Then* (3) *holds and is sharp.*

The technique used involves finding a marked Newton diagram for the quotient $\frac{\|f(z)\|^2-1}{\|z\|^2-1}$ with positive and negative coefficients marked.

Spherical equivalence is not the only natural notion of equivalence for proper holomorphic maps. A natural obstruction in many problems in complex analysis is topological, therefore it is natural to study the topology of the space of proper maps between balls. We say two maps $f\colon \mathbb{B}_n \to \mathbb{B}_{N_1}$ and $g\colon \mathbb{B}_n \to \mathbb{B}_{N_2}$ are *homotopic in dimension* N if there exists a continuous map $H(z,t) = H_t(z)$, $H\colon \mathbb{B}_n \times [0,1] \to \mathbb{B}_N$ such that H is continuous, H_t is a proper holomorphic map, $\iota_1 \circ f = H_0$, and $\iota_2 \circ g = H_1$, where $\iota_j\colon \mathbb{B}_{N_j} \to \mathbb{B}_N$ is the linear embedding. It is also natural to restrict the regularity of the maps involved. So we say that f and g are *homotopic through rational maps* if in addition H_t is rational for every t. Given a large enough N (in particular if $N = N_1 + N_2$), any two maps f and g as above are homotopic by so-called juxtaposition:

$$H_t(z) = tf \oplus \sqrt{1-t^2}\, g. \tag{4}$$

Many years ago D'Angelo noticed that the juxtaposition of the identity and the so-called Whitney map gives a family of spherically inequivalent mappings. By proving that given a fixed t_0, the set of parameters t such that H_t is spherically equivalent to H_{t_0} is closed, D'Angelo and I have been able to prove the following generalization:

Theorem 2.3 (D'Angelo-Lebl [8])**.** *Suppose H_t is a homotopy of proper rational maps between balls. If H_0 and H_1 are not spherically equivalent, then H_t contains uncountably many spherically inequivalent maps.*

A natural question is: how many equivalence classes exist for each pair (n, N)? Any proper map of $\mathbb{B}_1$ to $\mathbb{B}_1$ is a Blaschke product and so homotopic to z^d for some unique positive integer d. Therefore when $n = N = 1$, there are countably many equivalence classes. As mentioned above, a proper map of $\mathbb{B}_n$ to $\mathbb{B}_n$, $n \geq 2$ is an automorphism and therefore homotopic to the identity. Thus for $n = N > 1$, there is exactly one equivalence class.

Faran [10] proved that a proper map $f\colon \mathbb{B}_2 \to \mathbb{B}_3$, smooth up to $\partial\mathbb{B}_2$, is spherically equivalent to

$$\begin{aligned} &(z,w) \mapsto (z,w,0), \\ &(z,w) \mapsto (z,zw,w^2), \\ &(z,w) \mapsto (z^2,\sqrt{2}\,zw,w^2), \\ &(z,w) \mapsto (z^3,\sqrt{3}\,zw,w^3). \end{aligned} \tag{5}$$

We proved that the maps in (5) are all in distinct homotopy classes (through rational maps) in dimension 3. They are, however, all homotopic to the identity in dimension 5. The number of equivalence classes for the pair $(n, N) = (2, 3)$ is therefore exactly 4. In general we have:

Theorem 2.4 (D'Angelo-Lebl [8])**.** *Let S denote the set of homotopy classes in target dimension N of proper rational mappings $f\colon \mathbb{B}_n \to \mathbb{B}_N$. If $n \geq 2$, then S is a finite set.*

Define the hyperquadric

$$Q(a,b) = \{(z,w) \in \mathbb{C}^a \times \mathbb{C}^b : \|z\|^2 - \|w\|^2 = 1\}. \tag{6}$$

A linear fractional transformation normalizes $Q(a, b)$ so that $a > b$. If $f\colon M \to Q(A, B)$ is a CR map whose image lies in a hyperplane (affine space) W, that is $f(U) \subset W \cap Q(A, B)$. Then for some k, $\tilde{A}$, $\tilde{B}$ and an affine change of coordinates we have $W \cap Q(A, B) \cong Q(\tilde{A}, \tilde{B}) \times \mathbb{C}^k$. Therefore f is a direct sum of $f_1 U \to Q(\tilde{A}, \tilde{B})$ and an arbitrary $f_2\colon U \to \mathbb{C}^k$. Therefore Therefore, it is natural to consider only those maps whose image does not lie in a hyperplane.

Given M, the natural question is to find the set of (A, B) such that a map from M to $Q(A, B)$ exists. Hyperquadrics can be thought of as flat models, and this question is the analogue of Nash embedding. In the CR case, most manifolds do not admit an embedding into any $Q(A, B)$. For those that do, such as the algebraic manifolds, the set of (A, B) is the so-called CR complexity.

Theorem 2.5 (D'Angelo-Lebl [7])**.** *Let $n \geq 2$. There exists a K such that whenever $A + B \geq K$, $(A > 1)$ there exists a CR map $f\colon S^{2n-1} = Q(n, 0) \to Q(A, B)$ such that the image of f is not contained in a complex hyperplane. When $Q(A, B)$ is not equivalent to a sphere, f can be taken to be rational of arbitrary degree.*

The sphere case differs considerably from the hyperquadric case. A result such as the above theorem does not hold if the source is a hyperquadric not equivalent to a sphere, i.e. $b = 0$. Baouendi-Huang [2] proved that if $f\colon Q(a, b) \to Q(A, b)$ is a CR map with $a > b \geq 1$ and $A \geq b > 1$, then f is equivalent to a linear embedding. Later Baouendi-Ebenfelt-Huang [1] proved that $f\colon Q(a, b) \to Q(A, B)$ is a CR map with $a > b \geq 1$ and $A \geq B > 1$ and $B < 2b - 1$, then f maps to a complex hyperplane.

Theorem 2.6 (Grundmeier-Lebl-Vivas [14])**.** *Let $a \geq 2$, $b \geq 1$, and $a > b$. Let $U \subset Q(a, b)$ be a connected open set and $f\colon U \to Q(A, B)$ be a real-analytic CR map such that $f(U)$ does not lie in a complex hyperplane then*

$$A \leq C(a,b,B), \tag{7}$$

where $C = C(a, b, B)$ is a constant depending only a, b, and B.

The proof involves translating the problem into a problem in commutative algebra and applying Green's hyperplane restriction theorem [13]. As long as A and B are comparable and sufficiently large we obtain nontrivial CR maps. In fact as long as $A + B$ is large enough, and

$$\frac{B-b+3}{A} \geq \frac{b}{a} \qquad \text{and} \qquad \frac{A-b+2}{B+1} \geq \frac{b}{a}, \tag{8}$$

then we constructed rational maps $f\colon Q(a, b) \to Q(A, B)$. The sharp value $C(a, b, B)$ in (7) is not known, although appealing to the construction it must go to infinity as B goes to infinity.

There is a computational, and thus experimental, aspect of this problem. Many questions, can be answered computationally. In [24] Daniel Lichtblau (Wolfram Research) and I proved new results about sharp monomial CR maps from S^3 to S^{2N-1} and we have used these results together with independent computer code to classify all such maps up to degree $d = 17$. For this we have used *Mathematica*, my own mathematics software package Genius [23], and hand-tuned C code. the computations. I extended the computations up to degree $d = 21$ in [21].

Recently, Grundmeier and I [15] have extended the techniques of [14] to show that the initial monomial ideal of the map and its quotient are invariants of rational maps between spheres and hyperquadrics. If we only look at the span of the generators rather than the entire ideal, the technique extends beyond the algebraic, that is rational case. The initial monomial space of a vector subspace of germs of holomorphic functions at a point is an invariant under germs of biholomorphisms. The advantage of initial monomial ideal techniques is that they are readily computable via computer algebra systems.

3. Levi-flat hypersurfaces

Pseudoconvexity is the complex variables analogue of convexity. An interesting degenerate case is when a real hypersurface is pseudoconvex from both sides. It is then called *Levi-flat*.

Let $H \subset \mathbb{C}^N$ be a singular codimension-one real-analytic local subvariety (a real-hypervariety). Denote by H^* the set of nonsingular points of top dimension and by H_s the set of points where H is not a real-analytic submanifold. We say that H is a Levi-flat *real-hypervariety* if H^* is Levi-flat. With this definition it is natural to study the singularity of the topological relative closure $\overline{H^*}$. We define Levi-flat for higher codimensional manifolds, as an intersection of Levi-flat hypersurfaces in general position. With this definition we consider complex submanifolds to be Levi-flat.

Theorem 3.1 (Lebl [20])**.** *Let $H \subset \mathbb{C}^N$ be a local Levi-flat real-hypervariety. Then the singular set $(\overline{H^*})_s$ is Levi-flat near points where it is a CR real-analytic submanifold. Furthermore, if $(\overline{H^*})_s$ is a generic manifold, then $(\overline{H^*})_s$ is a generic Levi-flat manifold of dimension $2N-2$.*

An open question is if H has a Levi-flat stratification. That is, is there a stratification of H into real-analytic Levi-flat submanifolds. To this end I have begun recently to study the CR singularities of higher codimension submanifolds that are either Levi-flat or at least contained in a Levi-flat hypersurface. More on this in the next section.

As in the case of complex algebraic varieties, it is convenient to study algebraic Levi-flat hypervarieties in complex projective space. In high enough dimension all Levi-flat hypervarieties are singular [29]. The leaves of Levi-flat hypersurfaces are complex subvarieties if they are compact. The celebrated theorem of Chow says that any complex subvariety of $\mathbb{P}^n$ is necessarily algebraic. For Levi-flats I proved the following analogue of Chow's theorem.

Theorem 3.2 (Lebl [18])**.** *Let $H \subset \mathbb{P}^n$, $n \geq 2$, be an irreducible Levi-flat hypervariety with infinitely many compact leaves, such that locally H is defined by a meromorphic function. Then H is semialgebraic and contained in a pullback of a real-algebraic curve in $\mathbb{C}$ by a rational function.*

The compactness of the leaves is not enough to guarantee algebraicity. That is, the hypotheses cannot be weakened significantly.

Theorem 3.3 (Lebl [22])**.** *There exists a Levi-flat real-hypervariety $H \subset \mathbb{P}^2$ with all leaves complex hyperplanes, such that H is not contained in any proper real-algebraic subvariety of $\mathbb{P}^2$.*

Levi-flat hypersurfaces are defined by functions satisfying a complex Monge-Ampére type equation. An interesting question is to study a boundary value problem for this partial differential equation. That is, when can we take a compact codimension-two real submanifold and find a smooth Levi-flat hypersurface with this given boundary. In dimension 2 the question has been heavily studied. In dimension 3 and higher, there have been few results (essentially only [9]).

See [16] for the full bibliography. I have studied the case of real-analytic boundary, both locally and globally. Locally, I have classified [16] all possible real analytic CR submanifolds that are boundaries of a smooth Levi-flat hypersurface. In suitable coordinates (z, w), they are given by $\{\operatorname{Im} w_1 = f(z, \bar{z}, \bar{w}), \operatorname{Im} w_2 = 0\}$. Unless the boundary itself is Levi-flat, the hypersurface must be real analytic and unique. We also have the following global regularity and uniqueness result. Note that a generic compact submanifold of codimension 2 has only isolated CR singularities.

Theorem 3.4 (Lebl [17])**.** *Let $M \subset \mathbb{C}^N$, $N \geq 3$, be a compact real analytic submanifold of codimension 2. Suppose there exists a compact connected C^∞ Levi-flat hypersurface H with boundary, such that $\partial H = M$. If the CR singularities of M are isolated, then $H \setminus M$ is real analytic. Further, H is the unique compact connected Levi-flat C^∞ hypersurface with boundary M.*

4. Higher codimension CR submanifolds

The above results suggest the study of higher codimensional submanifolds. CR singular submanifolds of codimension 2 were first studied in $\mathbb{C}^2$ by E. Bishop [3], who found that such nondegenerate submanifolds are locally of the form

$$w = z\bar{z} + \lambda(z^2 + \bar{z}^2) + O(3), \tag{9}$$

for $\lambda \in [0, \infty]$ (where $\lambda = \infty$ means $w = z^2 + \bar{z}^2 + O(3)$). As mentioned before, dimension higher than 2 is not well understood yet.

With Xianghong Gong [12], we studied the normal form for CR singular Levi-flat submanifolds of codimension 2 in $\mathbb{C}^{n+1}$, $n \geq 2$. Locally such manifolds are of the form $w = \rho(z, \bar{z})$ for some ρ with $d\rho = 0$, where $(z, w) \in \mathbb{C}^n \times \mathbb{C} = \mathbb{C}^{n+1}$. The quadratic form can be normalized to:

$$\begin{aligned}
&(\mathrm{A}.k) \quad && w = \bar{z}_1^2 + \bar{z}_2^2 + \cdots + \bar{z}_k^2 + O(3), k = 1, \ldots, n, \\
&(\mathrm{B}.\lambda) \quad && w = |z_1|^2 + \lambda\bar{z}_1^2 + O(3),\ \lambda \geq 0, \\
&(\mathrm{C}.0) \quad && w = \bar{z}_1 z_2 + O(3), \\
&(\mathrm{C}.1) \quad && w = \bar{z}_1 z_2 + \bar{z}_1^2 + O(3).
\end{aligned} \tag{10}$$

The types C.x do not appear in dimension 2, and they are the only types stable under small perturbation. We proved that such submanifolds are in fact locally images of $\mathbb{R}^2 \times \mathbb{C}^{n-1}$:

Theorem 4.1. *Suppose $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, is a real-analytic Levi-flat CR singular submanifold of type C.1 or C.0. Then there exists a nonsingular real-analytic foliation on M extending the Levi-foliation on M_{CR}, and therefore, a CR real-analytic mapping $F \colon U \subset \mathbb{R}^2 \times \mathbb{C}^{n-1} \to \mathbb{C}^{n+1}$ such that F is a diffeomorphism onto $F(U) = M \cap U'$, for some neighbourhood U' of 0.*

We studied images of CR manifolds in general in [25].

Finally, we found a complete formal normal form for submanifolds of type C.1 in dimension 3, in particular showing that such submanifolds have infinitely many biholomorphic invariants.

We turn to submanifolds which are not completely flat, but which lie in a nonsingular Levi-flat. The most studied is the elliptic case (e.g. $0 \leq \lambda < 1/2$ in $n = 1$). With Alan Noell and Sivaguru Ravisankar, we studied extension of CR functions into the Levi-flat hypersurface. Suppose we have M given by

$$M : w = \sum_{j=1}^{n} \bigl(|z_j|^2 + \lambda_j(z_j^2 + \bar{z}_j^2)\bigr) + E(z, \bar{z}), \tag{11}$$

where $0 \leq \lambda_j < \frac{1}{2}$, and E is $O(3)$, smooth, and real-valued. The Levi-flat hypersurface H whose boundary is M is

$$H : \begin{cases} \operatorname{Re} w \geq \sum_{j=1}^{n} \bigl(|z_j|^2 + \lambda_j(z_j^2 + \bar{z}_j^2)\bigr) + E(z, \bar{z}), \\ \operatorname{Im} w = 0. \end{cases} \tag{12}$$

Any nondegenerate holomorphically flat elliptic M can be put into that form locally. The set of λ_j are biholomorphic invariants that are analogues to the Bishop invariant above.

Theorem 4.2 (Lebl-Noell-Ravisankar [26]). *Suppose H and M are closed submanifolds of $U = \{(z,w) \in \mathbb{C}^n \times \mathbb{C} : \|z\| < \delta_z, |w| < \delta_w\}$ given by* (11) *and* (12)*, (i.e. M is nondegenerate, holomorphically flat, and elliptic), $\delta_z, \delta_w > 0$ are small enough, and suppose $n > 1$. Suppose $f \colon M \to \mathbb{C}$ is smooth and f is a CR function on M_{CR}.*

Then there exists a function $F \in C^\infty(H)$ such that F is CR on $H \setminus M$ and $F|_M = f$. Furthermore, F has a formal power series at 0 in z and w. If M and f are real-analytic, then F is a restriction of a holomorphic function defined in a neighborhood of H in $\mathbb{C}^{n+1}$ (in particular F is real-analytic).

When $n = 1$ the theorem still holds under an additional necessary hypothesis that the function extend along leaves. The extension is obtained using the Bochner-Hartogs theorem (using Bochner-Martinelli integral kernel). However, the main point of the theorem is the regularity not the extension. The local theorem then leads to the gobal version of Bochner-Hartogs for $\mathbb{C}^n \times \mathbb{R}$:

Theorem 4.3 (Lebl-Noell-Ravisankar [26]). *Suppose $\Omega \subset \mathbb{C}^n \times \mathbb{R}$ is a bounded domain with smooth boundary, $n > 1$, and all CR singularities of $\partial\Omega$ are nondegenerate and elliptic. Suppose $f \colon \partial\Omega \to \mathbb{C}$ is smooth and f is a CR function on $(\partial\Omega)_{CR}$.*

Then there exists a function $F \in C^\infty(\overline{\Omega})$ such that F is CR on Ω and $F|_{\partial\Omega} = f$. Furthermore, if $\partial\Omega$ and f are real-analytic, then F is a restriction of a holomorphic function defined in a neighborhood of $\overline{\Omega}$ in $\mathbb{C}^{n+1}$ (in particular F is real-analytic).

The theorem again holds when $n = 1$ under additional hypothesis. Neither theorem (that is, the regularity) holds if the singularity is degenerate. Furthermore, as the extension can only be smooth along the real direction, we can only expect a holomorphic extension in the real-analytic case. If M is an image of a CR submanifold as in [25] then no extension is possible, even in the real-analytic case. This suggests that the existence of the extension is a non-generic phenomenon and one that works under certain nondegeneracy conditions in the holomorphically flat case where we extend along leaves. In the absence of flatness, extension cannot be expected in general.

The above theorem provides existence of the Levi-flat Plateau problem studied by Dolbeault-Tomassini-Zaitsev in case the boundary is an image of a boundary $\partial\Omega$ as above under a CR map.

5. Future plans

All the results mentioned suggest fertile ground for further work. For CR maps of spheres and hyperquadrics, the conjectured degree bound is still open for rational mapping and developing new techniques is needed to attack this problem. It remains my goal to prove this conjecture. There has been much recent work in extending rigidity of the type done by Baouendi-Ebenfelt-Huang to manifolds other than hyperquadrics, and I have been working with several coauthors in this direction. For singularities of Levi-flat hypersurfaces, my goal is to prove a stratification theorem similar to complex analytic varieties. A more short term goal is to understand the CR singular sets of higher codimension Levi-flat submanifolds.

Finally the plan is to generalize the CR extension results from the last section. It appears that in the real-analytic case at least, ellipticity is not necessary for local extension, under additional nondegeneracy conditions. We are currently working on this conjecture and have results in special cases.

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