Rigidity of CR maps of hyperquadrics

Jiří Lebl
joint work with Dusty Grundmeier and Liz Vivas

Department of Mathematics, University of Wisconsin-Madison

July 2012

Setup

Let $M \subset \mathbb{C}^n$ and $M' \subset \mathbb{C}^N$ be real submanifolds.

$F \colon M \to M'$ is *CR* if it satisfies tangential Cauchy-Riemann equations.

Setup

Let $M \subset \mathbb{C}^n$ and $M' \subset \mathbb{C}^N$ be real submanifolds.

$F \colon M \to M'$ is *CR* if it satisfies tangential Cauchy-Riemann equations.

Goal

Classify CR maps from M to M'.

Setup

Let $M \subset \mathbb{C}^n$ and $M' \subset \mathbb{C}^N$ be real submanifolds.

$F\colon M \to M'$ is *CR* if it satisfies tangential Cauchy-Riemann equations.

Goal

Classify CR maps from M to M'.

This is very hard.

So perhaps we can ask the question in some specific scenario.

Spheres

We could study spheres. That is

$$S^{2n-1} = \{z \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^2 = \|z\|^2 = 1\}$$

Consider CR maps $F\colon S^{2n-1} \to S^{2N-1}$, with $n \geq 2$.

Spheres

We could study spheres. That is

$$S^{2n-1} = \{z \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^2 = \|z\|^2 = 1\}$$

Consider CR maps $F\colon S^{2n-1} \to S^{2N-1}$, with $n \geq 2$.

The smallest N' such that $F(S^{2n-1})$ lies in an N'-dimensional affine space is the *embedding dimension* of F.

If $F(S^{2n-1}) \not\subset H$ for any affine complex hyperplane $H \subset \mathbb{C}^N$, then we'll say F has *minimal target dimension* ($N = N'$).

Embedding dimensions for maps of S^{2n-1} to S^{2N-1}

Which embedding dimensions are possible?

Embedding dimensions for maps of S^{2n-1} to S^{2N-1}

Which embedding dimensions are possible?

No maps possible if $N < n$.

Embedding dimensions for maps of S^{2n-1} to S^{2N-1}

Which embedding dimensions are possible?

No maps possible if $N < n$.

If $n = N$ then the only map is an automorphism of the unit ball (a linear fractional map) (Poincaré '07, Pinčuk '75).

Embedding dimensions for maps of S^{2n-1} to S^{2N-1}

Which embedding dimensions are possible?

No maps possible if $N < n$.

If $n = N$ then the only map is an automorphism of the unit ball (a linear fractional map) (Poincaré '07, Pinčuk '75).

Theorem (Webster '79)

If $n \geq 3$ there is no C^3 smooth CR map of spheres with embedding dimension $N = n + 1$.

Embedding dimensions for maps of S^{2n-1} to S^{2N-1}

Which embedding dimensions are possible?

No maps possible if $N < n$.

If $n = N$ then the only map is an automorphism of the unit ball (a linear fractional map) (Poincaré '07, Pinčuk '75).

Theorem (Webster '79)

If $n \geq 3$ there is no C^3 smooth CR map of spheres with embedding dimension $N = n + 1$.

Dor ('90) constructed a continuous CR map with embedding dimension $N = n + 1$.

Gaps for maps $F\colon S^{2n-1} \to S^{2N-1}$

No CR maps of spheres with minimal target dimension for the following "gaps":

n	Target dimension	Regularity
$n > 2$	$n < N < 2n-1$	real-analytic (Faran '86) C^{N-n+1} (Forstnerič '86, Cima-Suffridge '90) C^2 (Huang '99)
$n > 4$	$2n < N < 3n-3$	C^3 (Huang-Ji-Xu '06)
$n > 7$	$3n < N < 4n-6$	C^3 (Huang-Ji-Yin '12)

Conjectured kth gap is $kn < N < (k+1)n - \frac{k(k+1)}{2}$.

For $n = 2$ there are no gaps.

Large codimension

Theorem (D'Angelo, L. '09)

Let $n \geq 2$. $\exists$ an M such that $\forall N \geq M$ there exists a polynomial CR map $F\colon S^{2n-1} \to S^{2N-1}$ with minimal target dimension.

So everything is possible for the sphere (beyond a certain point).

Large codimension

Theorem (D'Angelo, L. '09)

Let $n \geq 2$. $\exists$ an M such that $\forall N \geq M$ there exists a polynomial CR map $F\colon S^{2n-1} \to S^{2N-1}$ with minimal target dimension.

So everything is possible for the sphere (beyond a certain point).

The sphere is defined by a positive definite form. What about surfaces defined by nondegenerate forms.

Hyperquadrics

Define

$$Q(a,b) \overset{def}{=} \left\{ z \in \mathbb{C}^{a+b} \ : \ \sum_{j=1}^{a} |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 = 1 \right\}$$

Note that

$$S^{2n-1} = Q(n,0)$$

Hyperquadrics

Define

$$Q(a,b) \overset{def}{=} \left\{ z \in \mathbb{C}^{a+b} \;:\; \sum_{j=1}^{a} |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 = 1 \right\}$$

Note that

$$S^{2n-1} = Q(n,0)$$

1) $a \geq 1$.
2) Levi-form of $Q(a,b)$ is of signature $(a-1,b)$.

Starting with a sphere

Theorem (D'Angelo, L. '11)

Let $n \geq 2$. $\exists M$ such that $\forall A, B$ with $A \geq 1$, $B \geq 0$, $A + B \geq M$, there exists a rational CR map $F \colon S^{2n-1} \to Q(A, B)$ with minimal target dimension.

Minimal target dimension is the same idea: $F(S^{2n-1})$ not contained in an affine complex hyperplane.

So really everything is possible when starting from a sphere (if we go far enough out).

CR maps between hyperquadrics

Suppose $F\colon U \subset Q(a,b) \to Q(A,B)$ is CR.

CR maps between hyperquadrics

Suppose $F\colon U \subset Q(a,b) \to Q(A,B)$ is CR.

When $Q(a,b)$ is not equivalent to a sphere, it is enough to consider real-analytic CR maps by a theorem of Lewy.

CR maps between hyperquadrics

Suppose $F\colon U \subset Q(a,b) \to Q(A,B)$ is CR.

When $Q(a,b)$ is not equivalent to a sphere, it is enough to consider real-analytic CR maps by a theorem of Lewy.

$Q(a,b) \cong Q(b+1, a-1)$ by a linear fractional map. So always assume that $a > b$ and $A > B$. Then $Q(a,b)$ is not equivalent to a sphere when $b \geq 1$.

Minimal target dimension

Let $F\colon U \subset Q(a,b) \to Q(A,B)$ be CR.

Minimal target dimension

Let $F\colon U \subset Q(a,b) \to Q(A,B)$ be CR.

If $F(U) \subset H$, where $H \subset \mathbb{C}^{A+B}$ is an affine complex hyperplane, then after an affine change of variables on the target side we have a map:

$$\widetilde{F}\colon U \to Q(A',B') \times \mathbb{C}^k$$

For some k, some $A' \leq A$, and $B' \leq B$.

Minimal target dimension

Let $F\colon U \subset Q(a,b) \to Q(A,B)$ be CR.

If $F(U) \subset H$, where $H \subset \mathbb{C}^{A+B}$ is an affine complex hyperplane, then after an affine change of variables on the target side we have a map:

$$\widetilde{F}\colon U \to Q(A',B') \times \mathbb{C}^k$$

For some k, some $A' \leq A$, and $B' \leq B$.

Example: Let F be

$$(z_1, z_2, z_3) \mapsto (z_1, z_2, \varphi(z), z_3, \varphi(z))$$

takes $Q(2,1)$ to $Q(3,2)$ (φ is arbitrary CR function). $H = \{w_3 = w_5\}$. Changing coordinates we obtain:

$$(z_1, z_2, z_3) \mapsto (z_1, z_2, z_3, \varphi(z), 0)$$

Note $Q(3,2) \cap H$ is equivalent to $Q(2,1) \times \mathbb{C}$.

Super-rigidity

Let

$$F \colon U \subset Q(a,b) \to Q(A,B)$$

be a real-analytic CR map. Suppose that $a > b \geq 1$, $A > B \geq 1$. Further suppose that F has minimal target dimension.

Super-rigidity

Let

$$F\colon U \subset Q(a,b) \to Q(A,B)$$

be a real-analytic CR map. Suppose that $a > b \geq 1$, $A > B \geq 1$. Further suppose that F has minimal target dimension.

Theorem (Baouendi-Huang '05)

If $b = B$, then $a = A$ and F is equivalent (via an LFT) to the identity.

Super-rigidity

Let

$$F\colon U \subset Q(a,b) \to Q(A,B)$$

be a real-analytic CR map. Suppose that $a > b \geq 1$, $A > B \geq 1$. Further suppose that F has minimal target dimension.

Theorem (Baouendi-Huang '05)

If $b = B$, then $a = A$ and F is equivalent (via an LFT) to the identity.

Theorem (Baouendi-Ebenfelt-Huang '09)

If $B < 2b$, then $a = A$, $b = B$ and F is equivalent to the identity.

Failure of super-rigidity

Let $(z, w) \in \mathbb{C}^a \times \mathbb{C}^b$. Then

$$(z, w) \mapsto (z_1, \ldots, z_{a-1}, z_a z_1, \ldots, z_a^2, z_a w_1, \ldots, z_a w_b, w_1, \ldots, w_b)$$

takes $Q(a, b)$ to $Q(2a - 1, 2b)$.

Rigidity

Theorem (Grundmeier, L., Vivas, '11)

Let $a > b \geq 1$*. Let* $U \subset Q(a,b)$ *be a connected open set and* $F\colon U \to Q(A,B)$ *be a real-analytic CR map with minimal target dimension, then*

$$A \leq N(a,b,B),$$

where $N(a,b,B)$ *is a constant depending only* a*,* b*, and* B*.*

Stability

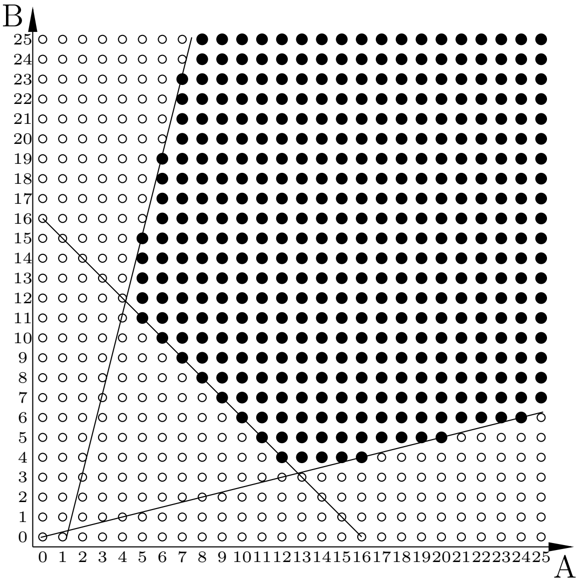
Theorem (Grundmeier, L., Vivas, '11)

Suppose $a > b \geq 1$, then there exists an N such that if $A + B \geq N$, and

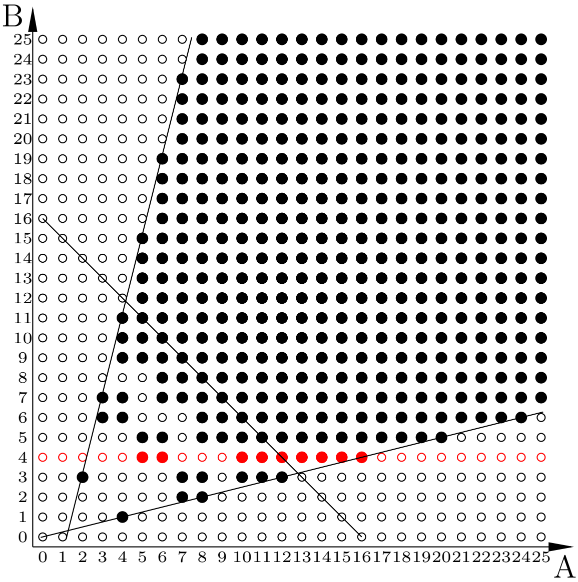
$$\frac{B - b + 3}{A} \geq \frac{b}{a} \qquad \text{and} \qquad \frac{A - b + 2}{B + 1} \geq \frac{b}{a},$$

then there exists a rational CR map $F \colon Q(a, b) \to Q(A, B)$ whose image does not lie in an affine complex hyperplane.

Picture is worth a thousand words $Q(4,1)$



Picture is worth a thousand words $Q(4, 1)$



Hermitian forms

Let r be a real-analytic function. Write

$$r(z,\bar{z}) = \|f(z)\|^2 - \|g(z)\|^2$$

for holomorphic Hilbert-space valued maps $f\colon \mathbb{C}^n \to \mathbb{C}^A$ and $g\colon \mathbb{C}^n \to \mathbb{C}^B$ with linearly independent components. Allow A and B to be ∞. (See D'Angelo '93)

Hermitian forms

Let r be a real-analytic function. Write

$$r(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2$$

for holomorphic Hilbert-space valued maps $f : \mathbb{C}^n \to \mathbb{C}^A$ and $g : \mathbb{C}^n \to \mathbb{C}^B$ with linearly independent components. Allow A and B to be ∞. (See D'Angelo '93)

Define

$$\text{rank } r = A + B \qquad \text{signature pair of } r = (A, B)$$

The decomposition non-unique, but A and B are well-defined.

Hermitian forms

Let r be a real-analytic function. Write

$$r(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2$$

for holomorphic Hilbert-space valued maps $f\colon \mathbb{C}^n \to \mathbb{C}^A$ and $g\colon \mathbb{C}^n \to \mathbb{C}^B$ with linearly independent components. Allow A and B to be ∞. (See D'Angelo '93)

Define

$$\text{rank } r = A + B \qquad \text{signature pair of } r = (A, B)$$

The decomposition non-unique, but A and B are well-defined.

Note that it looks like we are plugging $f \oplus g$ into the defining equation of a hyperquadric.

Matrix of coefficients

Assuming r is defined near 0 we can write

$$r(z, \bar{z}) = \langle C\mathcal{Z}, \mathcal{Z} \rangle$$

where $\mathcal{Z} = (1, z_1, z_2, \ldots, z_1^2, z_1 z_2, \ldots)$ is the vector of all monomials and C is a (formal) Hermitian matrix (called the *matrix of coefficients*).

Matrix of coefficients

Assuming r is defined near 0 we can write

$$r(z, \bar{z}) = \langle C\mathcal{Z}, \mathcal{Z} \rangle$$

where $\mathcal{Z} = (1, z_1, z_2, \ldots, z_1^2, z_1 z_2, \ldots)$ is the vector of all monomials and C is a (formal) Hermitian matrix (called the *matrix of coefficients*).

After possibly rescaling r, C is a Hermitian trace-class operator on ℓ^2 of rank $A + B$, and A positive and B negative eigenvalues.

Matrix of coefficients

Assuming r is defined near 0 we can write

$$r(z, \bar{z}) = \langle C\mathcal{Z}, \mathcal{Z} \rangle$$

where $\mathcal{Z} = (1, z_1, z_2, \ldots, z_1^2, z_1 z_2, \ldots)$ is the vector of all monomials and C is a (formal) Hermitian matrix (called the *matrix of coefficients*).

After possibly rescaling r, C is a Hermitian trace-class operator on ℓ^2 of rank $A + B$, and A positive and B negative eigenvalues.

We obtain f and g in $r = \|f\|^2 - \|g\|^2$ by diagonalizing C.

Hermitian forms: example (finite dimensional)

$$5z_1z_2\bar{z}_1\bar{z}_2 - 3z_2^2\bar{z}_2^2 + 2z_1^2 + 2\bar{z}_1^2 + z_1\bar{z}_1 =$$

$$= \begin{bmatrix} 1 & \bar{z}_1 & \bar{z}_2 & \bar{z}_1^2 & \bar{z}_1\bar{z}_2 & \bar{z}_2^2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ z_1 \\ z_2 \\ z_1^2 \\ z_1z_2 \\ z_2^2 \end{bmatrix}$$

$$= \left|\sqrt{5}\,z_1z_2\right|^2 + \left|1 + z_1^2\right|^2 + \left|z_1\right|^2 - \left|\sqrt{3}\,z_2^2\right|^2 - \left|1 - z_1^2\right|^2$$

Key ingredient

Let $G_{m,n}$ be the affine Grassmanian (affine complex m-planes in $\mathbb{C}^n$)

Theorem (Grundmeier, L., Vivas, '11)

Let $n \geq 2$ and let $1 \leq m \leq n-1$. Let $r \colon \Omega \subset \mathbb{C}^n \to \mathbb{R}$ be a nonzero real-analytic function (Ω connected and "small enough"). If

$$\max_{L \in G_{m,n}} \operatorname{rank} r|_L < \infty.$$

Then $\operatorname{rank} r < \infty$.
Moreover, $\exists$ $R_{m,n} \colon \mathbb{N} \to \mathbb{N}$ such that for all such r

$$\operatorname{rank} r \leq R_{m,n} \left(\max_{L \in G_{m,n}} \operatorname{rank} r|_L \right).$$

Actually not quite enough

The hypothesis "*for all* $L \in G_{m,n}$" is not good enough.

Actually not quite enough

The hypothesis "*for all* $L \in G_{m,n}$" is not good enough.

Let $\mathcal{L} \subset G_{m,n}$ be a *generic subset* (not contained in any complex subvariety of $G_{m,n}$).

We show that if r is *positive semi-definite* ($B = 0$), then

$$\operatorname{rank} r \leq R_{m,n} \left(\max_{L \in \mathcal{L}} \operatorname{rank} r|_L \right) .$$

When looking only at $\mathcal{L}$ then r must be positive semi-definite for any bound to hold.

Interesting consequence

Suppose $f\colon U \subset \mathbb{C}^n \to \mathbb{C}^N$ is holomorphic and fix $m < n$.

If for each affine m-plane L intersecting U the set $f(U \cap L)$ lies in an affine M-plane in $\mathbb{C}^N$, then $f(U)$ lies in an affine $R_{m,n}(M)$-plane.

Interesting consequence

Suppose $f\colon U \subset \mathbb{C}^n \to \mathbb{C}^N$ is holomorphic and fix $m < n$.

If for each affine m-plane L intersecting U the set $f(U \cap L)$ lies in an affine M-plane in $\mathbb{C}^N$, then $f(U)$ lies in an affine $R_{m,n}(M)$-plane.

Here also a generic set of L will do as well.

Idea of proof of the rigidity theorem

If $a > b \geq 1$, then $Q(a, b)$ contains a generic set $\mathcal{L}$ of affine b-planes.

Idea of proof of the rigidity theorem

If $a > b \geq 1$, then $Q(a, b)$ contains a generic set $\mathcal{L}$ of affine b-planes.
Let $f \oplus g$ map some neighbourhood $U \subset Q(a, b)$ to $Q(A, B)$ where $A + B$ is the minimal target dimension.

Idea of proof of the rigidity theorem

If $a > b \geq 1$, then $Q(a, b)$ contains a generic set $\mathcal{L}$ of affine b-planes. Let $f \oplus g$ map some neighbourhood $U \subset Q(a, b)$ to $Q(A, B)$ where $A + B$ is the minimal target dimension. Look at

$$r(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2$$

f maps to $\mathbb{C}^A$ and g maps to $\mathbb{C}^B$.

Idea of proof of the rigidity theorem

If $a > b \geq 1$, then $Q(a, b)$ contains a generic set $\mathcal{L}$ of affine b-planes. Let $f \oplus g$ map some neighbourhood $U \subset Q(a, b)$ to $Q(A, B)$ where $A + B$ is the minimal target dimension. Look at

$$r(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2$$

f maps to $\mathbb{C}^A$ and g maps to $\mathbb{C}^B$.
As $r = 1$ on $U \subset Q(a, b)$, then for every L in $\mathcal{L}$ that also intersects U

$$1 = r|_L = \|f|_L\|^2 - \|g|_L\|^2$$

Idea of proof of the rigidity theorem

If $a > b \geq 1$, then $Q(a, b)$ contains a generic set $\mathcal{L}$ of affine b-planes. Let $f \oplus g$ map some neighbourhood $U \subset Q(a, b)$ to $Q(A, B)$ where $A + B$ is the minimal target dimension. Look at

$$r(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2$$

f maps to $\mathbb{C}^A$ and g maps to $\mathbb{C}^B$.
As $r = 1$ on $U \subset Q(a, b)$, then for every L in $\mathcal{L}$ that also intersects U

$$1 = r|_L = \|f|_L\|^2 - \|g|_L\|^2$$

In other words, $\|f|_L\|^2 = \|g|_L\|^2 + 1$. The rank of $\|g|_L\|^2$ is bounded by B and hence rank of $\|f|_L\|^2$ is bounded by $B + 1$.

Idea of proof of the rigidity theorem

If $a > b \geq 1$, then $Q(a, b)$ contains a generic set $\mathcal{L}$ of affine b-planes. Let $f \oplus g$ map some neighbourhood $U \subset Q(a, b)$ to $Q(A, B)$ where $A + B$ is the minimal target dimension. Look at

$$r(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2$$

f maps to $\mathbb{C}^A$ and g maps to $\mathbb{C}^B$.
As $r = 1$ on $U \subset Q(a, b)$, then for every L in $\mathcal{L}$ that also intersects U

$$1 = r|_L = \|f|_L\|^2 - \|g|_L\|^2$$

In other words, $\|f|_L\|^2 = \|g|_L\|^2 + 1$. The rank of $\|g|_L\|^2$ is bounded by B and hence rank of $\|f|_L\|^2$ is bounded by $B + 1$.
As $\|f\|^2$ is positive semi-definite, the rank of $\|f\|^2$ (which is A) is bounded by some function of B. Which is exactly what we wanted.

Idea of proof of the rigidity theorem

If $a > b \geq 1$, then $Q(a, b)$ contains a generic set $\mathcal{L}$ of affine b-planes. Let $f \oplus g$ map some neighbourhood $U \subset Q(a, b)$ to $Q(A, B)$ where $A + B$ is the minimal target dimension. Look at

$$r(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2$$

f maps to $\mathbb{C}^A$ and g maps to $\mathbb{C}^B$.
As $r = 1$ on $U \subset Q(a, b)$, then for every L in $\mathcal{L}$ that also intersects U

$$1 = r|_L = \|f|_L\|^2 - \|g|_L\|^2$$

In other words, $\|f|_L\|^2 = \|g|_L\|^2 + 1$. The rank of $\|g|_L\|^2$ is bounded by B and hence rank of $\|f|_L\|^2$ is bounded by $B + 1$.
As $\|f\|^2$ is positive semi-definite, the rank of $\|f\|^2$ (which is A) is bounded by some function of B. Which is exactly what we wanted.
QED!

Proof of stability

Define $Q(a, b)$ by

$$s(z, \bar{z}) = \sum_{j=1}^{a} |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 = 1$$

Suppose $r = 1$ on $Q(a, b)$, where

$$r(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2$$

i.e. $f \oplus g$ takes $Q(a, b)$ to $Q(A, B)$.

Proof of stability

Define $Q(a, b)$ by

$$s(z, \bar{z}) = \sum_{j=1}^{a} |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 = 1$$

Suppose $r = 1$ on $Q(a, b)$, where

$$r(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2$$

i.e. $f \oplus g$ takes $Q(a, b)$ to $Q(A, B)$.

Then for an arbitrary holomorphic function φ

$$r_1 = \|f\|^2 - \|g\|^2 + |\varphi|^2 (s - 1)$$

generically adds a positive and $b + 1$ negative eigenvalues.

Proof of stability

Define $Q(a, b)$ by

$$s(z, \bar{z}) = \sum_{j=1}^{a} |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 = 1$$

Suppose $r = 1$ on $Q(a, b)$, where

$$r(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2$$

i.e. $f \oplus g$ takes $Q(a, b)$ to $Q(A, B)$.

Then for an arbitrary holomorphic function φ

$$r_1 = \|f\|^2 - \|g\|^2 + |\varphi|^2 (s - 1)$$

generically adds a positive and $b + 1$ negative eigenvalues.

$$r_2 = \|f\|^2 - \|g\|^2 + |\varphi|^2 (1 - s)$$

generically adds $b + 1$ positive and a negative eigenvalues.

Continued...

Let $f = (f', f_A)$.

$$r_3 = \|f'\|^2 + |f_A|^2 s - \|g\|^2$$

generically adds $a - 1$ positive and b negative eigenvalues.

$$r_4 = \|f'\|^2 + \frac{|f_A|^2}{2} + \frac{|f_A|^2 s}{2} - \|g\|^2$$

generically adds a positive and b negative eigenvalues.

Continued...

Let $f = (f', f_A)$.

$$r_3 = \|f'\|^2 + |f_A|^2 s - \|g\|^2$$

generically adds $a - 1$ positive and b negative eigenvalues.

$$r_4 = \|f'\|^2 + \frac{|f_A|^2}{2} + \frac{|f_A|^2 s}{2} - \|g\|^2$$

generically adds a positive and b negative eigenvalues.

By variations on the above obtain maps to:
$Q(A+a, B+b+1)$, $Q(A+a, B+b)$, $Q(A+a-1, B+b+1)$, $Q(A+a-1, B+b)$.

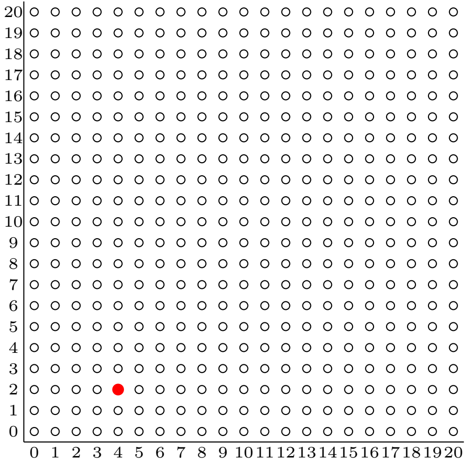
And also to:
$Q(A+b+1, B+a)$, $Q(A+b+1, B+a-1)$, $Q(A+b, B+a)$, $Q(A+b, B+a-1)$

Proof by picture

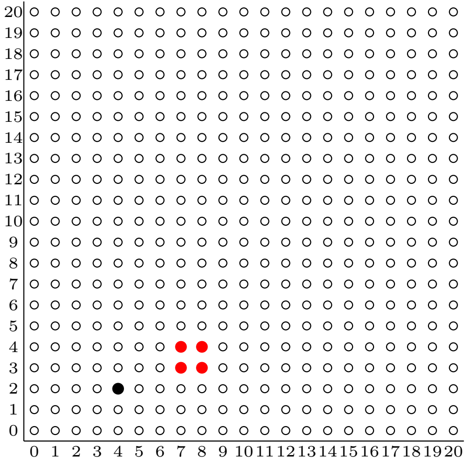
In the following pictures the B axis (vertical) is shifted by one for symmetry.

We show the construction of maps $Q(4,1) \to Q(A, B-1)$.

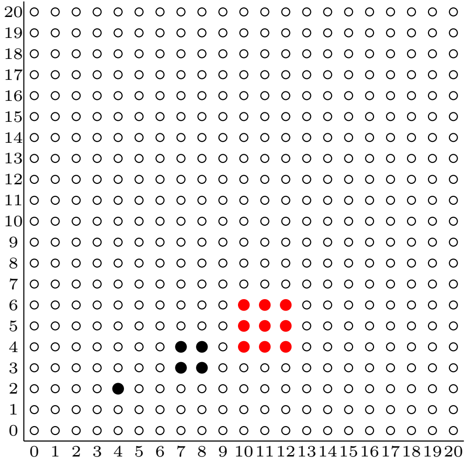
Proof by picture



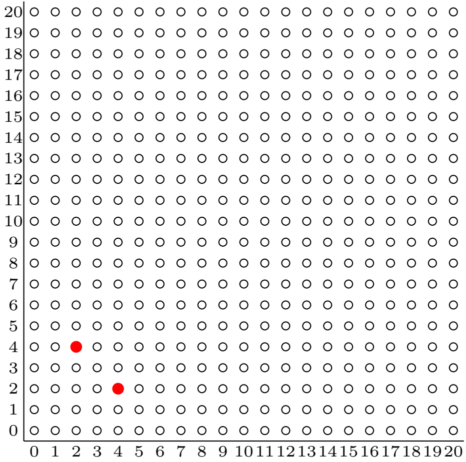
Proof by picture



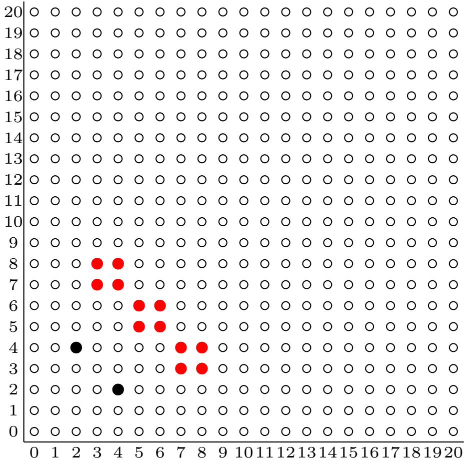
Proof by picture



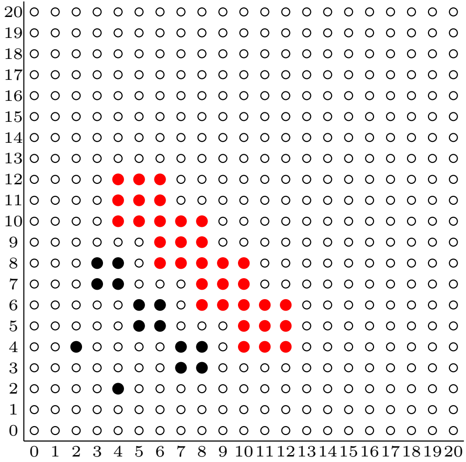
Proof by picture



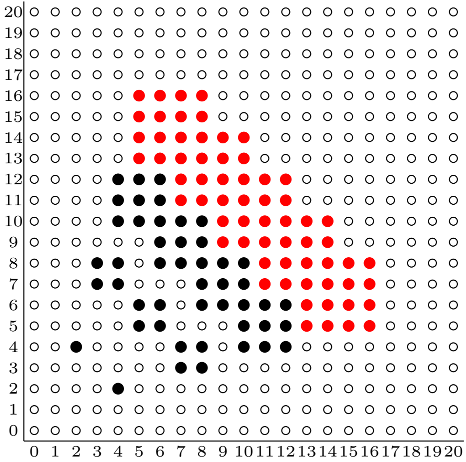
Proof by picture



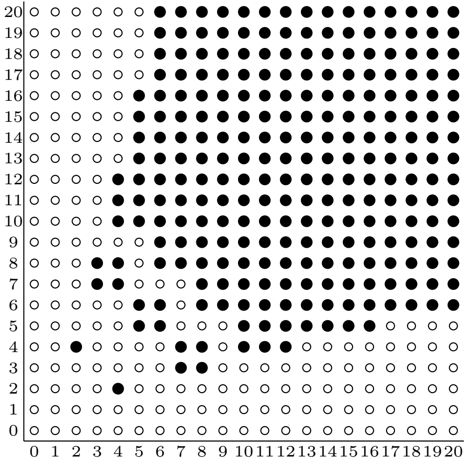
Proof by picture



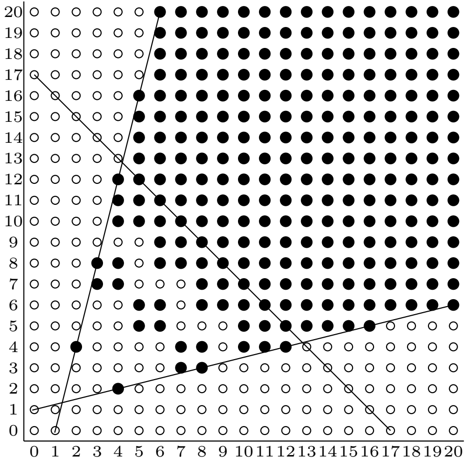
Proof by picture



Proof by picture



Proof by picture



Infinitely many dimensions

There exist strictly pseudoconvex real-analytic compact hypersurfaces that cannot be embedded via a real-analytic map into a sphere of any finite dimension (Forstnerič '86).

Every such hypersurface embeds (via a real-analytic map) into a sphere in ℓ^2 (Lempert '90).

Infinitely many dimensions

There exist strictly pseudoconvex real-analytic compact hypersurfaces that cannot be embedded via a real-analytic map into a sphere of any finite dimension (Forstnerič '86).

Every such hypersurface embeds (via a real-analytic map) into a sphere in ℓ^2 (Lempert '90).

What about non-pseudoconvex Levi-nondegenerate hypersurfaces?

Infinitely many dimensions

There exist strictly pseudoconvex real-analytic compact hypersurfaces that cannot be embedded via a real-analytic map into a sphere of any finite dimension (Forstnerič '86).

Every such hypersurface embeds (via a real-analytic map) into a sphere in ℓ^2 (Lempert '90).

What about non-pseudoconvex Levi-nondegenerate hypersurfaces?

$$Q(\infty, b) := \left\{ z \in \ell^2 : -\sum_{j=1}^{b} |z_j|^2 + \sum_{j=b+1}^{\infty} |z_j|^2 = 1 \right\},$$

$$Q(\infty, \infty) := \left\{ z \in \ell^2 : \sum_{j=1}^{\infty} \left(|z_{2j-1}|^2 - |z_{2j}|^2 \right) = 1 \right\}.$$

Note $Q(\infty, 0)$ is the unit sphere in ℓ^2.

Infinitely many dimensions

For every real-analytic hypersurface there exists a CR map into some $Q(A, B)$ if we allow A and B to be infinite (using holomorphic decomposition, D'Angelo '93).

Infinitely many dimensions

For every real-analytic hypersurface there exists a CR map into some $Q(A, B)$ if we allow A and B to be infinite (using holomorphic decomposition, D'Angelo '93).

Corollary (Grundmeier, L., Vivas, '11)

Let $\infty > a > b \geq 1$. Let $U \subset Q(a, b)$ be a connected open set and $f \colon U \to Q(\infty, B)$, where $B \in \mathbb{N}_0 \cup \{\infty\}$, be a real-analytic CR mapping such that $f(U)$ is not contained in any complex hyperplane of ℓ^2. Then $B = \infty$.

Indefinite Levi-form

$$r(z,\bar{z}) = e^{|z_1+1|^2+|z_2|^2} - e - |z_3|^2$$

has signature pair $(\infty, 2)$ and

$$r(z,\bar{z}) = 2e\,\mathrm{Re}\,z_1 + |z_2|^2 - |z_3|^2 + \text{higher order terms.}$$

So $M = \{r = 0\}$ has indefinite Levi-form at the origin.

Indefinite Levi-form

$$r(z,\bar{z}) = e^{|z_1+1|^2+|z_2|^2} - e - |z_3|^2$$

has signature pair $(\infty, 2)$ and

$$r(z,\bar{z}) = 2e\,\mathrm{Re}\, z_1 + |z_2|^2 - |z_3|^2 + \text{higher order terms.}$$

So $M = \{r = 0\}$ has indefinite Levi-form at the origin.

We obtain

$$f\colon M \to Q(\infty, 1)$$

whose image is not contained in a hyperplane.